Equivalence of Generalized Joint Signal Representations of Arbitrary Variables

Akbar M. Sayeed, Student Member, IEEE, and Douglas L. Jones, Member, IEEE

Abstract—Joint signal representations (JSR’s) of arbitrary variables generalize time-frequency representations (TFR’s) to a much broader class of nonstationary signal characteristics. Two main distributional approaches to JSR’s of arbitrary variables have been proposed by Cohen and Baraniuk. Cohen’s method is a direct extension of his original formulation of TFR’s, and Baraniuk’s approach is based on a group theoretic formulation; both use the powerful concept of associating variables with operators.

One of the main results of the paper is that despite their apparent differences, the two approaches to generalized JSR’s are completely equivalent. Remarkably, the JSR’s of the two methods are simply related via axis warping transformations, with the broad implication that JSR’s with radically different covariance properties can be generated efficiently from JSR’s of Cohen’s method via simple pre- and post-processing. The development in this paper, which is illustrated with examples, also illuminates other related issues in the theory of generalized JSR’s. In particular, we derive an explicit relationship between the Hermitian operators in Cohen’s method and the unitary operators in Baraniuk’s approach, thereby establishing the relationship between the two types of operator correspondences.

I. INTRODUCTION

TIME-FREQUENCY representations (TFR’s) provide a description of signal characteristics jointly in terms of time and frequency by measuring the time-varying spectral energy in the signal [1], [2]. Whereas TFR’s are well-suited for representing a fairly rich class of nonstationary signal characteristics, they are inadequate in other situations, such as those involving a nonlinear chirping behavior. To encompass a wider variety of signal characteristics, recently, there has been significant interest in the development of generalized joint signal representations (JSR’s), which analyze signals in terms of physical quantities other than time and frequency [3]–[5], [1], [6]–[11]. The wavelet transform and generalizations are the best known, which analyze signals in terms of time and scale content [12], [3]–[5], [7].

Owing to the recent interest in JSR’s, there has been substantial progress in the development of a general theory for generalized JSR’s with respect to arbitrary variables. The most comprehensive theory to date is due to Cohen [1], [6], who has developed the generalization first proposed by Scully and Cohen [13] in direct extension of Cohen’s original method for generating TFR’s [14]. Baraniuk and Jones developed a general procedure for generating a wide class of JSR’s from existing ones via the principle of unitary equivalence [9], [15]. More recently, Baraniuk has proposed a more general theory that is similar in principle to Cohen’s general method but is based on group theoretic arguments [16], [17]. Other generalizations have also been proposed by Hlawatsch and Bülcskei [18], [19] and Sayeed and Jones [20], which characterize generalized JSR’s that are covariant to certain unitary transformations and complement the distributional approaches developed by Cohen and Baraniuk. Our main interest in this paper is in the general methods developed by Cohen [1], [6] and Baraniuk [16], [17].

Fundamental to both Cohen’s and Baraniuk’s methods is the idea of associating variables of interest with operators. Cohen’s method associates variables with Hermitian (self-adjoint) operators, whereas Baraniuk’s approach associates them with families of unitary operators that are unitary representations of certain one-parameter groups. Both methods use analogs of the characteristic function operator method that was originally introduced by Cohen [14] in which the characteristic function of the joint distribution is first computed via the characteristic function operator, and then the distribution is recovered from it. Baraniuk’s method appears to be more general than Cohen’s since the latter can be recovered from the former by basing the construction on the group of real numbers. Cohen’s method, on the other hand, is generally more attractive computationally since it is based on the Fourier transform as opposed to arbitrary group transforms. Moreover, in some situations, it is more natural to associate unitary operators with variables, as is done in Baraniuk’s approach, whereas in others, the Hermitian operator correspondence is more straightforward. Thus, an understanding of the relationship between these two

1In general, there are infinitely many possibilities (correspondence rules) for the characteristic function operator, resulting in infinitely many JSR’s.

2As pointed out by one of the reviewers, and as will be explained later in the paper, certain correspondence rules for the characteristic function operator cannot be defined in the general group setting of Baraniuk’s approach. Cohen’s approach, on the other hand, does not have this drawback, and our development of the equivalence results in this paper enables us to define the equivalents of all the correspondence rules in Cohen’s approach for Baraniuk’s recipe. This yields an extension to Baraniuk’s method that encompasses all the different correspondence rules, and throughout the paper, “Baraniuk’s method” implicitly refers to this “extended Baraniuk’s method.”
powerful methods is essential from the viewpoint of both the theory and application of generalized JSR’s.

One of the main results of this paper is that despite the apparent differences between Cohen’s and Baraniuk’s methods, the two approaches to generalized JSR’s are completely equivalent. We prove that there is a one-to-one and onto mapping that relates the JSR’s constructed via Baraniuk’s method to those generated by Cohen’s approach. In addition to explicitly characterizing this mapping, we also derive the explicit relationship between the unitary and Hermitian operators used in the two methods.

Remarkably, the JSR’s in the two methods are simply related via *axis warping* transformations, implying that group transforms in Baraniuk’s method, which are not computationally efficient in general, may be replaced with the Fourier transform as in Cohen’s approach. The broad implication of the results is that JSR’s with radically different characteristics can be generated efficiently from JSR’s in Cohen’s method by simple pre- and post-processing.

The development in this paper also allows us to address some related issues that have not been addressed adequately in existing treatments. An example is the relationship between Hermitian and unitary operator correspondences, which is fundamental to the understanding of JSR’s of arbitrary variables. Using Stone’s theorem [21] and the notion of *duality*, we characterize the relationship between the two types of correspondences.

In the next section, after providing a brief primer on relevant operator- and group-theoretic concepts, we describe the two distributional approaches to generalized JSR’s. In Section III, we briefly outline an equivalent description of Baraniuk’s method in terms of Hermitian operators that is useful in interpreting the main results presented in Section IV regarding the equivalence of the methods. Section V discusses some related issues in light of the presented results, and Section VI concludes the paper with a summary of the results and their implications.

II. PRELIMINARIES

In order to present the results of the paper, we need a description of the two methods for generalized JSR’s. For simplicity, we will consider joint distributions of two variables only; extension to multiple variables is straightforward. Table I specifies the notation that we adopt in the two methods. Dual operators are defined in the next section on background material.

A. Operator- and Group-Theoretic Background

Baraniuk’s approach is based on associating variables with parameterized unitary operators that are related to certain one-parameter locally compact (abelian) (LCA) groups. Let $G$ be the underlying one-parameter LCA group with group operation $\cdot$

One-parameter groups are necessarily abelian, and thus, the term “abelian” is somewhat redundant. However, we keep the notation LCA (as opposed to LC) because some of our discussion applies to arbitrary LCA groups as well. The restriction to one-parameter groups will be either explicit or clear from the context. All final results relating to joint distributions are based on one-parameter groups.

<table>
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<td>Dual Unitary Operators</td>
<td>$A_{x_i}^\alpha, B_{y_i}^\beta$</td>
<td>$K_{x_i}^\kappa, L_{y_i}^\lambda$</td>
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We will use the symbols $k, l$ for elements of $G$ (the variables in Baraniuk’s method; see Table I).

**Characters and the Dual Group:** A complex-valued function $\kappa$ on $G$ is called a character of $G$ if $|\kappa(k)| = 1$ for all $k \in G$ and if it satisfies the functional equation $\kappa(k \cdot l) = \kappa(k) \kappa(l)$, for all $k, l \in G$ [22]. Note that a character $\kappa$ maps the group $G$ into the complex unit circle. The set of all continuous characters of $G$ itself forms a one-parameter LCA group $\Gamma$, which is called the dual group of $G$, with the group operation $\circ$ defined by $(\kappa \circ \lambda)(k) \equiv \kappa(k)\lambda(k)$, $k \in G$, $\kappa, \lambda \in \Gamma$ [22]. Because of this duality, it is convenient to use the following symmetric notation for characters $\kappa(k) \equiv (k, \kappa)$. In case of ambiguity, we will explicitly use the group as the subscript; for example, $(k, \kappa)_G$.

**The Natural Signal Spaces and the Group Transform:** The natural signal space associated with the group $G$ is $\mathcal{H}_1 = L^2(G, d\mu_G)$, which is the space of square-integrable functions defined on $G$, where $\mu_G$ is the Haar measure associated with $G$, which is invariant to group translation, that is, $\mu_G(E \cdot k) = \mu_G(E)$, for all measurable sets $E \subset G$ and all $k \in G$. Similarly, the space associated with the dual group is $\mathcal{H}_2 = L^2(\Gamma, d\mu_\Gamma)$. The natural analog of the Fourier transform is the group transform $\mathbb{F}_G: \mathcal{H}_1 \rightarrow \mathcal{H}_2$, which is an isometry between $\mathcal{H}_1$ and $\mathcal{H}_2$ and defined as

$$\mathbb{F}_G(s)(\kappa) \equiv \int_G s(k)(k, \kappa)^* d\mu_G(k), \quad s \in \mathcal{H}_1$$

with the inverse transform $\mathbb{F}_G^{-1}: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ defined by

$$\mathbb{F}_G^{-1}(h)(k) \equiv \int_\Gamma h(K(k, \kappa)) d\mu_\Gamma(k), \quad h \in \mathcal{H}_2.$$
fundamental unitary representations of $G$—one defined over $\mathcal{H}_1$ and the other over $\mathcal{H}_2$—are the group translation operator $T^G_{l}: \mathcal{H}_1 \rightarrow \mathcal{H}_1$ defined as

$$ (T^G_{l}s)(k) = s(k \cdot l) \quad (3) $$

and the “diagonal” operator $\Lambda^G_{l}: \mathcal{H}_2 \rightarrow \mathcal{H}_2$ defined as

$$ (\Lambda^G_{l}h)(k) = (l, k)h(k) \quad (4) $$

which is simply multiplication by a character. The corresponding fundamental unitary representations of $\Gamma$ are $T^\Gamma_{l, \kappa}: \mathcal{H}_2 \rightarrow \mathcal{H}_2$ defined as

$$ (T^\Gamma_{l, \kappa}h)(k) = h(\kappa \circ l) \quad (5) $$

and the diagonal operator $\Lambda^\Gamma_{l, \kappa}: \mathcal{H}_2 \rightarrow \mathcal{H}_2$ given by

$$ (\Lambda^\Gamma_{l, \kappa}s)(k) = (l, \kappa, \lambda)^* s(k). \quad (6) $$

Note that for $G = \mathbb{R}$ and identifying it with time, the translation operator $T^G_{t}$ essentially reduces to the time-shift operator $T^\Gamma_{t, \kappa}(T^\Gamma_{t, \kappa}l)(t) \equiv s(t - \tau)$, and the diagonal operator $\Lambda^G_{t}$ reduces to the frequency-shift operator $F_{\theta}(F_{\theta}s)(\ell) \equiv e^{2\pi i \ell \theta} s(\ell)$. It is well known that the group translation operators and the diagonal operators satisfy the following fundamental relationships [22, 23]:

$$ \Lambda^G_{l} = IF_{\kappa}T^G_{l}IF_{\kappa}^{-1} \quad \text{and} \quad \Lambda^\Gamma_{l, \kappa} = IF_{\kappa}T^\Gamma_{l, \kappa}IF_{\kappa}^{-1} \quad (7) $$

which essentially state that group translation in one domain is equivalent to multiplication with characters in the dual domain.

**Spectral Representation of Operators and Energy Densities:** By Stone’s theorem [21], a group of unitary operators $K_{k}$ admits the spectral representation

$$ K_{k} = S_{K}^{-1} A_{k} S_{K} \quad (8) $$

where $S_{K}: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a unitary transform (isometry) determined by the (generalized) eigenfunctions of the operator $K_{k}$.

Note that $S_{K}$ “diagonalizes” $K_{k}$. Define a new unitary transform $S_{K^{0}}: \mathcal{H}_1 \rightarrow \mathcal{H}_1$ via

$$ S_{K^{0}} = IF_{\kappa}^{-1} S_{K}. \quad (9) $$

Using (7) and (8), we arrive at the following alternative representation for $K_{k}$:

$$ K_{k} = S_{K}^{-1} A_{K} S_{K} = S_{K}^{-1} T^G_{l} IF_{\kappa}^{-1} IF_{\kappa} T^\Gamma_{l, \kappa} S_{K} = S_{K}^{-1} T^G_{l} \Lambda^G_{K} S_{K} \quad (10) $$

From (8), it follows that the transform $S_{K}$ is $K$-invariant [16, 11] in the sense that $[(S_{K}K_{k}s)(\kappa)] = [(\Lambda^G_{K}s)(\kappa)] = [(S_{K}s)(\kappa)]$ for all $s \in \mathcal{H}_1$ and $k \in G$. On the other hand, from (10), it follows that the transform $S_{K}$ is $K$-covariant [16, 11] in the sense that $(S_{K}K_{k}s)(l) = (T^G_{k}s)(l \cdot k^{-1})$ for all $s \in \mathcal{H}_1$ and $k \in G$. That is, $[(S_{K}s)(\kappa)]$ is invariant to the effect of the operator $K_{k}$, whereas applying the operator $K_{k}$ to the signal produces a group translation of $k^{-1}$ in the signal representation $(S_{K}s)(\kappa)]$.

**Duality:** Define a new unitary operator $K_{k}^{0}: \mathcal{H}_1 \rightarrow \mathcal{H}_1$, $\kappa \in \Gamma$ via the spectral representation

$$ K_{k}^{0} = S_{K}^{-1} A_{k}^{0} S_{K} \quad (11) $$

which is a unitary representation of $\Gamma$ on $\mathcal{H}_1$. In fact, by Stone’s theorem, any unitary representation of $\Gamma$ on $\mathcal{H}_1$ admits a spectral representation of the form (11). It immediately follows that the transform $S_{K}$ is $K^{0}$-invariant, and using (7) and (11), it can be easily verified that $S_{K}$ is $K^{0}$-covariant, that is, $(S_{K}K_{k}^{0}s)(\lambda) = (S_{K}(\Lambda^{G}_{k}) \lambda \circ \kappa^{-1})$.

Given two groups of unitary operators $K_{k}$ and $L_{\lambda}$, there exist, again by Stone’s theorem [21], two unique Hermitian operators $\mathcal{K}$ and $L^{0}$, respectively, such that formally

$$ K_{k} = (k, \kappa) \quad \text{and} \quad L_{\lambda}^{0} = (L^{0}, \lambda)^* \quad (12) $$

and $\mathcal{K}$ and $L^{0}$ admit the spectral representations

$$ \mathcal{K} = S_{K}^{-1} A^{G} S_{K} = S_{K}^{-1} A^{\Gamma} S_{K} $$

and

$$ L^{0} = S_{L}^{-1} A^{G} S_{L} = S_{L}^{-1} A^{\Gamma} S_{L} \quad (13) $$

where the diagonal operators $A^{\Gamma}$ and $A^{G}$ are defined on appropriate subspaces of $\mathcal{H}_2$ and $\mathcal{H}_1$, respectively, as

$$ (A^{\Gamma}x)(\kappa) = kx(\kappa), \quad \kappa \in \Gamma \quad (14) $$

$$ (A^{G}x)(k) = kx(k), \quad k \in G \quad (15) $$

Note that both $\mathcal{K}$ and $L^{0}$ share the same eigenfunctions, that is, $S_{K} = S_{L}$, and similarly for $L^{0}$ and $L^{0}$. The diagonal operators $A^{\Gamma}$ and $A^{G}$ are Hermitian and are related to the parameterized diagonal unitary operators $A^{G}_{\kappa}$ and $\Lambda^{\Gamma}_{\kappa, \lambda}$, which have been defined in (4) and (6), as $A^{G}_{\kappa} = (k, \kappa)$ and $A^{\Gamma}_{\kappa} = (\lambda, \kappa, \lambda)^*$. We are now in a position to define the notion of dual operators.

**Definition—Dual Operators:** Let $K_{k} = S_{K}^{-1} A^{G}_{k} S_{K}$ and $L_{\lambda}^{0} = S_{L}^{-1} A^{G}_{L} S_{L}$ be unitary representations of $G$ and $\Gamma$, respectively, on $\mathcal{H}_1$, and let $\mathcal{K}$ and $L^{0}$ be the corresponding Hermitian operators defined by Stone’s theorem. Then, the operators $K_{k}(\kappa)$ and $L_{\lambda}^{0}(\lambda)$ are **dual** if the spectral families of $K_{k}(\kappa)$ and $L_{\lambda}^{0}(\lambda)$ are related by $S_{K} = IF_{\kappa}^{-1} S_{L} = (IF_{\kappa}^{-1} S_{L} S_{K})^{-1}$.

If $L_{\lambda}^{0}$ and $L^{0}$ are dual to $K_{k}$ and $\mathcal{K}$, we denote them by $K_{k}^{0}$ and $\mathcal{K}^{0}$. 

9Throughout the paper, we denote unitary representations of the dual group $\Gamma$ with the superscript $0$. The corresponding Hermitian operators defined by Stone’s theorem will be denoted similarly.

10The operators $A^{G}_{\kappa}$ and $\Lambda^{\Gamma}_{l, \kappa}$ are dual operators, and similarly, $A^{G}_{\kappa}$ and $\Lambda^{\Gamma}_{l, \kappa}$ are dual operators.
Example: Let \((G, \cdot) = (\mathbb{R}_+, \times)\), where \(\mathbb{R}_+ = (0, \infty)\) and \(\times\) denotes multiplication. One characterization of the dual group is \((\Gamma, \circ) = (\mathbb{R}_+, +)\) with the characters given by \((k, \xi) \equiv e^{2\pi k \xi \ln(b)}\). In this case, \(d\mu_G(k) = dk/k\), and \(d\mu_T(\xi) = d\xi\). The group Fourier transform is the Mellin transform \([24], [25]\)

\[
(F_{\mathbb{R}_+}, s)(\xi) \equiv \int_0^\infty s(k)e^{-j2\pi k \xi \ln(b)}dk/k.
\] (16)

B. Cohen’s Approach

Let \(a\) and \(b\) be two variables of interest; they could be time and scale, for example. In Cohen’s approach, the variables \(a\) and \(b\) are associated with appropriate Hermitian operators \(A\) and \(B\), respectively. The eigenfunctions of \(A\) and \(B\) define unitary signal representations \(S_A\) and \(S_B\), which yield the \(a\) and \(b\) representations of the signal. Ideally, the joint distribution \((Ps)(a,b)\) should satisfy the \(a\) and \(b\) marginals, that is, \([1]\)

\[
\int (Ps)(a,b)da = |(S_A s)(a)|^2,
\] (17)

\[
\int (Ps)(a,b)db = |(S_B s)(b)|^2.
\] (18)

The characteristic function \(M\) is defined as \([1]\)

\[
(Ms)(\alpha, \beta) = \int \int (Ps)(a,b)e^{2\pi i \alpha a e^{2\pi i \beta b}}dadb
\] (19)

and thus, the distribution \(P\) can be recovered from \(M\) as \([20]\)

\[
(Ps)(a,b) = \int \int (Ms)(\alpha, \beta)e^{-2\pi i \alpha a e^{-2\pi i \beta b}}d\alpha d\beta.
\] (20)

The key observation in Cohen’s method is that the characteristic function, being an average of \(e^{2\pi i \alpha a e^{2\pi i \beta b}}\), can be directly computed from the signal using a characteristic function operator \([1]\) \(M_{(\alpha, \beta)}\) corresponding to the function \(e^{2\pi i \alpha a e^{2\pi i \beta b}}\), an example being \(M_{(\alpha, \beta)} = e^{2\pi i \alpha A e^{2\pi i \beta B}}\), that is, \([11]\)

\[
(Ms)(\alpha, \beta) = \langle M_{(\alpha, \beta)} s, s \rangle = \int \langle M_{(\alpha, \beta)} s(t), s^*(t) \rangle dt.
\] (21)

However, in general, there are infinitely many ways to associate an operator with the function \(e^{2\pi i \alpha a e^{2\pi i \beta b}}\), and thus infinitely many corresponding joint distributions. Taken together, the different distributions generated by these infinitely many operator correspondences define the class of joint \(a\)-\(b\) distributions (the operator method). As a simple description of all the joint distributions, Cohen has proposed the kernel method, which implicitly assumes that all the different characteristic functions can be generated by weighting a particular one, say, \(M_0\), with a 2-D kernel \([1]\). For example, one formulation could be

\[
(Ms)(\alpha, \beta) = \phi(\alpha, \beta)(Ms_0)(\alpha, \beta) = \phi(\alpha, \beta)e^{2\pi i \alpha A e^{2\pi i \beta B} s, s}
\] (22)

where \(\phi\) is the 2-D kernel. Another formulation is given by the Weyl correspondence \([26], [1], [27], [28]\)

\[
M^W_{(\alpha, \beta)} = e^{2\pi i (\alpha A + \beta B)}.
\] (23)

C. Baraniuk’s Method

In Baraniuk’s method, variables are associated with operators that are unitary representations of a one-parameter LCA group, say \(G\). Suppose we are interested in the joint distributions of two variables associated with the groups of unitary operators \(K_k\) and \(L_l\), \(k, l \in G\). Baraniuk’s approach allows us to recover either the IED or CED marginal corresponding to an operator, that is, a joint distribution \(P\) satisfies

\[
\int (Ps)(u,v)d\mu(v) = |\mathcal{S}_K(u)|^2
\] (24)

\[
\int (Ps)(u,v)d\mu(u) = |\mathcal{S}_L(v)|^2
\] (25)

where the measure \(\mu\) is either \(\mu_G\) or \(\mu_I\), and \(\mathcal{S}_K\) is either the IED (K-IED) or CED (K-CED) (\(\mathcal{S}_L\) is defined similarly) \([16]\). In this approach as well, a modification of the characteristic function method is used. If the K-IED marginal is desired, define the unitary operator \(K_k = K_{\alpha}\), and if the CED marginal is desired, define \(K_{\alpha} = K_{\alpha}^o\), which is the dual operator of \(K_k\). Similarly, define \(L_l\) and \(L_{l^*}\). Since there are four different combinations, each corresponding to different pairs of marginals, we illustrate with a specific case parallel to \([16]\); other cases are obvious. Thus, suppose we are interested in K-CED and L-IED marginals. Then, the characteristic function is computed as

\[
(Ms)(\kappa, l) = \hat{\phi}(\kappa, l)(K_{\kappa}L_{l} s, s) = \hat{\phi}(\kappa, l)(K_{\kappa}L_{l} s, s)
\] (26)

and the distribution can be recovered (using \(F_G\)) as

\[
(\hat{P}(\hat{\phi})s)(k, \lambda) = \int \int (Ms)(\kappa, l)(k, \lambda)\cdot d\mu_T(\kappa)\cdot d\mu_G(l)
\] (27)

which yields the K-CED and L-IED marginals if \(\hat{\phi}(\kappa, 0) = 1\) for all \(\kappa \in \Gamma\), and \(\hat{\phi}(0, l) = 1\) for all \(l \in G\) \([16]\).

Note that this method also assumes that all the different characteristic functions can be generated from a particular one, namely, \((Ms)(\kappa, l) = \langle K_{\kappa}L_{l} s, s \rangle\), via a weighting kernel. However, this is not true in general \([29], [30]\). Moreover, in this method, as described in \([16]\), the choices for the characteristic function operator \(M_0\) are rather limited as compared with Cohen’s method; in particular, there is no

\[\text{Note:} A^o_\alpha = e^{2\pi i \alpha A} \text{ and } B^o_\beta = e^{2\pi i \beta B} \text{ are unitary representations of} (\mathbb{R}, +) \text{ on } L^2(\mathbb{R}, dx).\]
analog of Weyl correspondence as defined in (23). However, our development will enable us to define equivalents of different correspondence rules in Cohen’s method, such as Weyl correspondence, for Baraniuk’s recipe. This equivalent operator method in Baraniuk’s approach, based on different correspondence rules, yields an extended form of Baraniuk’s method. Moreover, by using different correspondence rules \( \tilde{\mathcal{M}} \), Baraniuk’s kernel method (26) and (27)) generates subsets of the entire class of distributions.

III. BARANIUK’S APPROACH IN TERMS OF HERMITIAN OPERATORS

Using Stone’s theorem, the unitary operators in Baraniuk’s method can also be uniquely associated with Hermitian operators, providing an interpretation of Baraniuk’s method in terms of Hermitian operators. In this section, we briefly describe this interpretation, which makes the comparison between Cohen’s and Baraniuk’s methods more explicit and helpful in understanding the mechanism of the equivalence results of next section.

We simply mimic Cohen’s general recipe (based on the group \((\mathbb{R}, +)\)) in the more general setting of LCA groups and show that it naturally leads to Baraniuk’s method. Since the natural signal spaces are \( L^2(G, d\mu_G) \) and \( L^2(\mathbb{R}, d\mu_l) \), the variables of interest can take on values in \( G \) or \( \mathbb{R} \) with the corresponding Hermitian operators of the form \( K_h \) and \( \mathcal{L} \), as defined in (13). Suppose we have two variables \( k \in G \) and \( \lambda \in \Gamma \) associated with operators \( K_h \) and \( \mathcal{L} \), and we are interested in the corresponding JSR’s \( \mathcal{P}(\psi)(k, \lambda) \). The natural definition of the characteristic function based on the group Fourier transform is

\[
(\hat{M})_h(k, \lambda) = \mathbb{F}_G^{-1} \cdots \mathbb{F}_G^{-1} \cdots \mathbb{F}_G^{-1} (\hat{\mathcal{P}}_h)(k, \lambda) = \int_G (\hat{\mathcal{P}}_h)(k, \lambda) e^{i\langle k, \lambda \rangle} d\mu_G(k) d\mu_{\mathcal{I}}(\lambda)
\]

where \( \hat{\mathcal{P}}_h \) is an arbitrary 2-D kernel. The corresponding JSR’s can then be recovered by inverting (28). The second equality in (29) shows the relationship with Baraniuk’s original approach via the unitary operators \( K_h \) and \( L_{\mathcal{I}} \). The resulting JSR \( \mathcal{P}(\psi)(k, \lambda) \) satisfies the following marginal:

\[
\int_\Gamma (\mathcal{P}(\psi)(k, \lambda)) d\mu_{\mathcal{I}}(\lambda) = \|S_{K_h} s(k)\|^2
\]

and only if \( \phi(k, 0) \equiv 1 \) for all \( k \in \Gamma \)

\[
\int_G (\mathcal{P}(\psi)(k, \lambda)) d\mu_G(k) = \|S_{L_{\mathcal{I}}} s(\lambda)\|^2
\]

and only if \( \phi(0, l) \equiv 1 \) for all \( l \in G \).

Note that if we take \( (G, \cdot) = (\mathbb{R}, +) \), we recover Cohen’s method outlined in Section II-B.

IV. EQUIVALENCE OF COHEN’S AND BARANIUK’S APPROACHES

In Section II, we described Cohen’s and Baraniuk’s approaches to generalized JSR’s. Baraniuk’s generalization, which is based on unitary representations of certain one-parameter LCA groups, is apparently broader than Cohen’s, since, as we saw in the previous section, Cohen’s method can be recovered as a special case by taking \( (G, \cdot) = (\mathbb{R}, +) \). In this section, we derive the main results of the paper that prove that despite the apparent differences between them, the two approaches are completely equivalent. An explicit relationship between the operators of the two methods is also characterized.

As mentioned in Section II-C, equivalents of certain correspondence rules in Cohen’s method, such as Weyl correspondence (23), cannot be directly described in Baraniuk’s method. Using the development in this section, we define the equivalent, in Baraniuk’s recipe, of any arbitrary correspondence rule in Cohen’s method. This yields an extended form of Baraniuk’s method that we prove to be equivalent to Cohen’s method.

The equivalence of the two methods stems from the fact that Baraniuk’s approach is implicitly based on a class of one-parameter LCA groups that are isomorphic to each other. The reason is that in Baraniuk’s method, different groups define different signal spaces, and thus, in order to construct joint distributions of variables belonging to different groups, it must be possible to relate the corresponding signal spaces. Moreover, in view of the fundamental importance of time and frequency variables (related to \((\mathbb{R}, +)\), the class of groups must contain \((\mathbb{R}, +)\) (the dual group of \((\mathbb{R}, +)\) is also \((\mathbb{R}, +)\)). It follows that if \( (G, \cdot) \) is the underlying group in Baraniuk’s construction, then there exist isomorphisms \( \psi : G \rightarrow \mathbb{R} \) (onto \mathbb{R} \) and \( \varphi : \Gamma \rightarrow \mathbb{R} \) (onto \mathbb{R} \), which are not unique in general. However, given \( \psi \), we can explicitly characterize a dual isomorphism \( \varphi \) satisfying certain conditions.
properties, as shown next. These isomorphisms play a central role in all the results of the paper.

A. Characterization of a Dual Isomorphism

Let \((G_1, \bullet)\) and \((G_2, *)\) be two isomorphic LCA groups, and let \((\Gamma_1, \circ)\) and \((\Gamma_2, \cdot)\) be the dual groups. Then, there exists an isomorphism \(\psi: G_1 \rightarrow G_2\) (onto \(G_2\)) such that \(\psi(k \bullet l) = \psi(k) * \psi(l)\) [22]. Without loss of generality, we assume that given \(\psi\), the Haar measures \(\mu_{G_1}\) and \(\mu_{G_2}\) are appropriately normalized so that [22]

\[
\mu_{G_1}(E) = \mu_{G_2}(\psi(E)) \quad \text{for all measurable sets } E \subset G_1.
\]

(32)

Similarly, without loss of generality, we assume that given the normalized measures \(\mu_{G_1}\) and \(\mu_{G_2}\), the dual Haar measures \(\mu_{G_1}\) and \(\mu_{G_2}\) are (individually) normalized so that the (corresponding) forward and inverse group Fourier transforms are symmetric without any additional factors [22]. With this setting, the following theorem, which is proved in the Appendix, explicitly characterizes a dual isomorphism \(\varphi: \Gamma_1 \rightarrow \Gamma_2\) and, thus, explicitly relates the characters of \(G_1\) and \(G_2\).

Theorem I: For each \(\kappa \in \Gamma_1\), define \(\varphi(\kappa) \in \Gamma_2\) as

\[
(m, \varphi(\kappa))_{G_2} \equiv (\psi^{-1}(m), \kappa)_{G_1}, \quad m \in G_2.
\]

(33)

Then, the functional equation (33) defines a continuous isomorphism \(\varphi: \Gamma_1 \rightarrow \Gamma_2\), which is onto \(\Gamma_2\) with a continuous inverse and satisfies

\[
\mu_{\Gamma_1}(E) = \mu_{\Gamma_2}(\varphi(E)) \quad \text{for all measurable sets } E \subset \Gamma_1.
\]

(34)

Note that (33) is well defined, that is, for each character \(\kappa \in \Gamma_1\), the right-hand side of (33) is well defined for all \(m \in G_2\) and defines a corresponding (continuous) [21] character of \(G_2\), which is denoted by \(\varphi(\kappa)\) on the left-hand side. Theorem I states that defining characters of \(G_2\) in this way characterizes the entire dual group \(\Gamma_2\) in terms of \(\Gamma_1\) and that the measures are preserved by the resulting isomorphism \(\varphi\).

Example (continued): Let \((G_1, \bullet) = (\mathbb{R}_+, \times)\) and \((G_2, *) = (\mathbb{R}_+, +) = (\Gamma_2, \cdot)\). One characterization of the characters of \(G_2\) is \((a, a) = e^{2\pi \alpha \kappa}, \ a, \alpha \in \mathbb{R}\). Since in this case \((\Gamma_1, \circ) = (\Gamma_2, \cdot) = (\mathbb{R}_+, +), it is clear that \(\varphi(\kappa) = \kappa\), which can also be inferred from (33) starting with \(\psi(k) = \ln(k)\)

\[
e^{j2\pi \alpha \varphi(\kappa)} = (\psi^{-1}(a), \kappa)_{\mathbb{R}_+} = e^{j2\pi \alpha \ln(e^\kappa)} = e^{j2\pi \alpha \kappa}
\]

for all \(\kappa, \ a \in \mathbb{R}\). (35)

B. Relationship Between the Signal Spaces

Now suppose that \((G, \bullet)\) is the underlying group in Baraniuk’s method, with \((\Gamma, \circ)\) being the dual group, and \(\mathcal{H}_1 = L^2(G, d\mu_G)\) the underlying signal space. Then, there exists an isomorphism \(\psi: G \rightarrow \mathbb{R}\), and let \(\varphi: \Gamma \rightarrow \mathbb{R}\) be characterized as in Theorem I. We use the following characterization for the characters of \((\mathbb{R}_+, +): (a, \alpha) = e^{j2\pi \alpha \kappa}\). Let \(K_k\) and \(L_l\) be two unitary operators in Baraniuk’s approach corresponding to the variables whose joint representations are desired. Recall that \(K_k\) and \(L_l\) are unitary representations of \(G\) on \(\mathcal{H}_1\).

The signal space in Cohen’s method is \(L^2(\mathbb{R}, dx)\), and the mapping that relates it to \(L^2(G, d\mu_G)\) is \(T_\psi: L^2(\mathbb{R}, dx) \rightarrow L^2(G, d\mu_G)\), which is defined as

\[
(T_\psi s)(k) = s(\psi(k)), \quad k \in G.
\]

(36)

\(T_\psi\) is an isometry from \(L^2(\mathbb{R}, dx)\) onto \(L^2(G, d\mu_G)\) since

\[
||T_\psi g||_{L^2_{\mu_1}} = \int_G |g(\psi(k))|^2 d\mu_G(k)
\]

\[
= \int_{\mathbb{R}} |g(x)|^2 d\mu_G(\psi^{-1}(x))
\]

\[
= \int_{\mathbb{R}} |g(x)|^2 dx = ||g||^2_{L^2_{\mu_2}}
\]

(37)

where the last equality follows from (32). The mapping \(T_\psi\) also relates the operators defined on \(\mathcal{H}_1\) to those defined on \(L^2(\mathbb{R}, dx)\). For example, \(K_k\) on \(L^2(G, d\mu_G)\) defines a corresponding unitary operator \(A_{x_k}\) on \(L^2(\mathbb{R}, dx)\) via

\[
A_{x_k} = T_\psi^{-1} K_k T_\psi
\]

(38)

which is a unitary representation of \((\mathbb{R}_+, +)\) on \(L^2(\mathbb{R}, dx)\) since for all \(a, b \in \mathbb{R}\)

\[
A_a A_b = T_\psi^{-1} K_{\psi^{-1}(a)} T_\psi K_{\psi^{-1}(b)} T_\psi = T_\psi^{-1} K_{\psi^{-1}(a+b)} T_\psi = A_{a+b}.
\]

(39)

Similarly, if we have a unitary representation of \(\Gamma\) on \(\mathcal{H}_1\), \(K_\kappa\), then the operator

\[
A_{\kappa} = T_\psi^{-1} K_{\psi^{-1}(\kappa)} T_\psi
\]

(40)

is also a representation of \((\mathbb{R}_+, +)\) (the dual group) on \(L^2(\mathbb{R}, dx)\). Given \(A_a\) and \(A_{\kappa}\), by Stone’s theorem [21], there exist unique Hermitian operators \(A\) and \(A^\circ\) defined on \(L^2(\mathbb{R}, dx)\) such that

\[
A_a = e^{j2\pi \alpha A} \equiv (a, A)_\mathbb{R}
\]

and

\[
A_{\kappa} = e^{-j2\pi \alpha A^\circ} \equiv (\kappa, A^\circ)_\mathbb{R}.
\]

(41)

20This implies that an isomorphism preserves the identity element, that is, \(\psi(0) = 0\).

21Recall from Section II-A that the dual group is the set of all continuous characters. Continuity of \(\varphi(k)\) (as a character) defined in (33) follows from the continuity of \(|k, \kappa|_{G_1}\) on \(G_1 \times \Gamma_1\) [22, p. 10] and the continuity of \(\psi^{-1}\) (see footnote 19).

22Due to the continuity of isomorphisms and inverse isomorphisms (see footnote 19), the continuity of unitary representations is preserved (see footnote 6).
Extended Baraniuk’s Method: Using the mapping \( T_{\psi} \), we can define the equivalent, in Baraniuk’s method, of any arbitrary correspondence rule in Cohen’s method. Let \( M_{(\alpha,\beta)} \) be any arbitrary (characteristic function) operator correspondence rule in Cohen’s method. Define the equivalent of \( M_{(\alpha,\beta)} \) in Baraniuk’s method (corresponding to K-CED and L-IED marginals) as

\[
M_{(\kappa,l)}(k,l) = T_{\psi}M_{(\kappa,\psi(l))}T_{\psi}^{-1}.
\]  

These operator correspondence rules yield an extended Baraniuk’s method that encompasses both the operator and kernel methods just as in Cohen’s method.

Example (continued): The mapping relating \( L^2(\mathbb{R}^2,\mu_{\mathbb{R}^2}) = L^2(\mathbb{R}_+,dk/k) \) and \( L^2(\mathbb{R},dx) \) is \( (T_{\psi}^l,\psi(l)) = s(l/k), k \in \mathbb{R}_+ \). The dilation operator \( (D_{\kappa,l},\kappa,l) \) is a unitary representation of \( (\mathbb{R_+},\times) \) (which is the group translation operator) on \( L^2(\mathbb{R},d\mu_{\mathbb{R}}) \) and gets mapped to the time-shift operator \( (T_{\alpha,l},\alpha) = s(\alpha-a\delta) \), which is the group translation operator on \( L^2(\mathbb{R},dx) \), via (38).

C. Equivalence Results

We are now in a position to prove the main result of the paper. The following theorem proves the equivalence of the two methods for the subsets of JSR’s generated, via the kernel method, by the correspondence rules of the form (22) and (26). Analogous arguments then yield the equivalence of the extended Baraniuk’s method and Cohen’s approach.24

Since there are four types of JSR’s in Baraniuk’s approach, corresponding to the choice of marginals, we characterize the equivalence for representations with one CED and one IED marginal; the equivalence for the remaining types can be readily inferred from the stated result.

Theorem 2: For each \( \hat{P} \) from Baraniuk’s class of JSR’s corresponding to operators \( K_{\kappa} \) and \( L_{\kappa} \) and yielding K-CED and L-IED marginals, there exists a \( P \) in a corresponding Cohen’s class (which is associated with a pair of Hermitian operators) of JSR’s (and vice versa) such that

\[
(\mathcal{P}(\hat{\phi}) s)(k,\lambda) = (P(\phi)T_{\psi}^{-1}s)(\psi(k),\phi(\lambda))
\]  

where \( (T_{\psi}^l,\psi(l)) \equiv s(\psi(k)) \)

is an isometry from \( L^2(\mathbb{R},dx) \) onto \( L^2(\mathbb{R},d\mu_{\mathbb{R}}) \), and the kernels are related as

\[
\hat{\phi}(\kappa,l) = \phi(\varphi(\kappa),\varphi(l)).
\]  

Moreover, the equivalence (43) is an isometry; that is, \( (\mathcal{P}(\hat{\phi}) s)(k,\lambda) = (V\mu_{\mathbb{R}}P(\phi)U\mu_{\mathbb{R}} s)(k,\lambda) \) (46)

where \( U_{\mathbb{R}} = T_{\psi}^{-1} \) and \( V_{\mathbb{R}} \), which is defined as

\[
(V_{\mathbb{R}} P)(k,\lambda) \equiv P(\psi(k),\varphi(\lambda))
\]  

is an isometry from \( L^2(\mathbb{R}^2, dx \times dx) \) onto \( L^2(G \times \Gamma, d\mu_{\mathbb{R}} \times d\mu_{\mathbb{R}}) \).

Proof: From (26) and (27), we see that \( \hat{P}(\phi) \) is given by

\[
(\mathcal{P}(\hat{\phi})s)(k,\lambda) = \int_G \int_{\Gamma} \hat{\phi}(\kappa,l)(K_{\kappa}L_{\lambda}s,s)(k,\lambda)(\kappa,l) d\mu_{\mathbb{R}}(\kappa)d\mu_{\mathbb{R}}(l),
\]  

where we have substituted \( K_{\kappa} = K_{\kappa} \) and \( L_{\lambda} = L_{\lambda} \) in (26) since K-CED and L-IED marginals are desired. Using (38) and (40), we get

\[
(\mathcal{P}(\hat{\phi})s)(k,\lambda) = \int_G \int_{\Gamma} \hat{\phi}(\kappa,l)(A_{\kappa}B_{\lambda}T_{\psi}^{-1}s,T_{\psi}^{-1}s) \times (k,\kappa)(\lambda,l) d\mu_{\mathbb{R}}(\kappa)d\mu_{\mathbb{R}}(l)
\]  

for some unitary operators \( A_{\kappa} \) and \( B_{\lambda}, \kappa, \lambda \in \mathbb{R}, \) which are unitary representations of \( (\mathbb{R},+) \) on \( L^2(\mathbb{R},dx) \). Making the substitutions \( l = \psi^{-1}(b) \) and \( \kappa = \varphi^{-1}(\alpha) \) in (49) yields

\[
(\mathcal{P}(\hat{\phi})s)(k,\lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\phi}(\kappa^{-1}(\alpha),\psi^{-1}(b))(A_{\kappa}B_{\lambda}T_{\psi}^{-1}s,T_{\psi}^{-1}s) \times (k,\kappa^{-1}(\alpha))(\lambda,\varphi^{-1}(\alpha)) d\mu_{\mathbb{R}}(\kappa^{-1}(\alpha)) d\mu_{\mathbb{R}}(\psi^{-1}(b))
\]  

where in the last equality, \( \phi \) is defined as in (45), and we used (41) and the fact that \( d\mu_{\mathbb{R}}(\psi^{-1}(b)) = db \) and \( d\mu_{\mathbb{R}}(\varphi^{-1}(\alpha)) = d\alpha \), which follows from (32) and (34) (Theorem 1). Finally, using the functional equation (33) in Theorem 1, which relates the characters of \( (G,\cdot) \) and \( (\mathbb{R},+) \), (50) becomes

\[
(\mathcal{P}(\hat{\phi})s)(k,\lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(a,b)(e^{-j2\pi\alpha\cdot\lambda} - j2\pi\alpha\cdot\lambda) d\alpha db
\]  

Comparing (51) with (22) and (20), we get the relation (43). Since we already know that \( T_{\psi} \) is an isometry, the only remaining thing to verify is that \( V_{\mathbb{R}} \) is an isometry, which simply follows from (32) and (34).

It is worth noting that the equivalence between the two approaches is simply based on axis warping transformations of the signal (via \( U_{\mathbb{R}} = T_{\psi} \)) and the JSR’s (via \( V_{\mathbb{R}} \)). Moreover, the axis transformations are simply the group isomorphisms \( \psi \) and \( \varphi \). This also makes it almost trivial to infer, from Theorem 2, the equivalence relationships corresponding to other choices of marginals in Baraniuk’s approach. First note that the signal warping transformation is always

23 We note that such warping transformations are employed in some specific classes of generalized JSR’s [8], [10]. For example, the hyperbolic class [8] is related to Cohen’s class of TFR’s [33], and the power class [10] is related to the affine class [5] via warping.
since the underlying signal space is $L^2(G, d\mu_G)$; only the axis warping of the JSR's changes. If a CED marginal is desired, the corresponding axis of the representation is warped by the group isomorphism $\psi$, whereas if an IED marginal is desired, the corresponding axis transformation is the dual isomorphism $\varphi$. Thus, given any group and any type of JSR in Baraniuk's construction, we need only determine the group isomorphisms to find the equivalent in Cohen's method. Moreover, the proof of Theorem 2 also defines, albeit somewhat implicitly, the corresponding operators of the equivalent Cohen's representation. The next theorem explicitly characterizes the relationship between the operators of the two methods. Recall the definitions of the diagonal operators $A^\varphi$, $A^\Gamma$, $A^\psi$, and $A^\sigma$ in Section II-A.

**Theorem 3:** Let $K_k$ and $L_l$ be the two unitary operators in Baraniuk's approach, whose JSR's, with K-CEP and L-IED marginals, are desired. Then, in the equivalent Cohen's class of JSR's, the Hermitian operator $A^\varphi$ corresponding to $K_k = S_K^{-1}A^\varphi S_K$ is given by

$$A^\varphi = S_K^{-1}A\sigma S_K$$  \hspace{1cm} (52)

where

$$\langle A \sigma \rangle (c) = cs(c), \quad c \in \mathbb{R}$$  \hspace{1cm} (53)

and

$$S_A = T_{\varphi}^{-1}S_K^{-1}T_{\psi}$$  \hspace{1cm} (54)

where $S_K$ is the transform determined by the eigenfunctions of $K_k = K_\sigma = S_K^{-1}A_\sigma S_K$, which is the dual operator of $K_k$. Similarly, the Hermitian operator $B$ corresponding to $L_l = S_L^{-1}A_l S_L$ is given by

$$B = S_L^{-1}A_B S_L$$  \hspace{1cm} (55)

where

$$S_B = T_{\varphi}^{-1}S_L^{-1}T_{\psi}$$  \hspace{1cm} (56)

**Proof:** The relationships (52) and (55) are essentially inferred from (50) in the proof of Theorem 2 via the relationships

$$A^\varphi = e^{-j2\pi c \alpha A^\varphi}$$  \hspace{1cm} (57)

$$B = e^{j2\pi n B}$$  \hspace{1cm} (58)

with $A^\varphi = T_{\varphi}^{-1}A_{\sigma}^{-1}(x)T_{\psi}$, and $B = T_{\varphi}^{-1}L_{\psi}^{-1}(b)T_{\psi}$. We first prove (52). We have

$$A^\varphi = T_{\varphi}^{-1}K_{\varphi}^{-1}(x)T_{\psi} = T_{\varphi}^{-1}S_K^{-1}A_\sigma^{-1}(x)S_K^{-1}T_{\psi} = T_{\varphi}^{-1}S_K^{-1}T_{\psi}^{-1}A_\sigma^{-1}(x)T_{\psi} = S_K^{-1}A_\sigma S_K$$  \hspace{1cm} (59)

where in the last equality, $(A_\sigma s)(c) = e^{-j2\pi c \alpha s}(c), \quad c \in \mathbb{R}$, and we used (54) and the fact that $T_{\varphi}^{-1}A_\sigma^{-1}(x)T_{\psi} = A_\sigma$, which readily follows from Theorem 1. Equation (59) can be rewritten in the form (57) with $A^\varphi$ exactly given by (52), and thus, by the uniqueness of the Hermitian operator defined by a group of unitary operators via Stone's theorem [21], $A^\varphi$ is given by (52) and (54).

Similarly, we have

$$B = T_{\psi}^{-1}L_{\psi}^{-1}(b)T_{\psi} = T_{\psi}^{-1}S_L^{-1}A_\Gamma^{-1}(b)S_L T_{\psi} = T_{\psi}^{-1}S_L^{-1}T_{\psi}^{-1}A_\varphi^{-1}(b)T_{\psi} = S_B^{-1}A_\varphi S_B$$  \hspace{1cm} (60)

where, in the last equality, we used (56) and the fact that $T_{\varphi}^{-1}A_\varphi^{-1}(b)T_{\psi} = A_\varphi$. Thus, by the same arguments as for $A^\varphi$, we see that $B$ is given by (55) and (56).

It should be noted that whatever the underlying group, the Hermitian and unitary operators are completely characterized by their (generalized) eigenfunctions since all the operators based on a particular group share the same diagonal operator (based on the eigenvalues) in their spectral representation. For Hermitian operators, the diagonal operator is of the form (14) or (15), and for unitary operators, it is essentially multiplication with a character of the underlying group (see (4) and (6)). Thus, to relate different operators based on different groups, we only need to relate their eigenfunctions, which is precisely what is done in Theorem 3. An alternative interpretation of (52) and (54), in terms of the operators of the Hermitian counterpart of Baraniuk's approach, is $A^\varphi = T_{\varphi}^{-1}A^\psi(K)T_{\psi}$, where $A^\varphi = S_K^{-1}A^\Gamma S_K$. Similarly, $B = T_{\psi}^{-1}A^\psi(L)T_{\psi}$, where $L = S_L^{-1}A^\Gamma S_L$.

Example (continued): Recall that $(G, \ast) = (\mathbb{R}_+, \ast)$, and the dilation operator $2^k_d s(k) \equiv s(k/2^k)$ is a unitary representation of $G$ on $L^2(\mathbb{R}_+, dk/k)$, which gets mapped to the time-shift operator $(T_\sigma s)(a) \equiv s(a - a')$ in $L^2(\mathbb{R}, da)$ (Cohen's method). This is precisely the reason why, using the correspondence rule of (26), the joint time-scale representations based on the T-CEP, D-CEP marginals or the T-IED, D-IED marginals are trivial [16]: $T_{\varphi}$ and $D_{\varphi}$, when represented in the same signal space, are identical! Working in the $L^2(\mathbb{R}_+, dk/k)$ space, the symmetric Wigner-like representation corresponding to T-CEP, D-IED marginals yields exactly the Q-distribution of Altes [25]. The corresponding representation in Cohen's method, using Theorem 2, is the Wigner distribution (WD) $W$, corresponding to the variables time and frequency, that is, $(Q s)(e^f, f) = (WT_{\psi}^{-1} s(t, f)$, where $(T_{\psi}^{-1} s(t)) = s(e^t)$, which is exactly the relationship derived in [25].

Note that in Baraniuk's method, the above example can also be construed, quite misleadingly though, as the joint representation based on a *single* (!) variable corresponding to the unitary operator $D_{\varphi}$. Just choose both the operators as $D_{\varphi}$ and design the representation to yield CED marginal in one case and the IED marginal in the other case. The reason for the apparent confusion is that when a CED marginal is desired in Baraniuk's method, the underlying operator is effectively the dual operator, which, as we argue in the next section using Stone's theorem, corresponds to a different variable (the dual variable to be precise). Thus, we are actually dealing with two operators, which are duals of each other, and, hence, two dual variables. This is also manifested in the fact that the corresponding Cohen's representation is the WD, which is based on the dual variables of time

26 which is associated with scale in [16] and [17].
and frequency [11]. This example illustrates the importance of precisely defining the operator correspondence principles and the relationship between Hermitian and unitary operator correspondences. Such issues are briefly discussed in the next section.

Example (continued): We can also illustrate the extended Baraniuk’s method in this example. In Cohen’s method, let $B = F$, which is the frequency Hermitian operator, and let $A = C \equiv \frac{1}{\sqrt{2\pi}}I - TF$, which is the “Mellin” Hermitian operator [11], [32]. The characteristic function operator for Weyl correspondence (23) is given by $M_{W}^{\alpha \beta} = e^{j2\pi(\alpha c + \beta s)} = e^{j2\pi e^{j\theta}/2\pi - j2\pi \alpha x}$. [32], [7], which yields the affine Wigner distribution $S_{W} = \frac{-j}{2\pi}\frac{d}{dt}s(t)$. (63)

By Stone’s theorem, $T$ and $F$ define two unitary operators given by

$$ F_{\theta} = e^{j2\pi \theta s} $$

and

$$ T_{\tau} = e^{j2\pi \tau} $$

Thus, the unitary operator associated with $T$ via Stone’s theorem is the frequency-shift operator, and the unitary operator associated with $F$ is the time-shift operator. Thus, mathematically, it is tempting to associate time with $F_{\theta}$ and frequency with $T_{\tau}$. However, intuition and physical interpretation dictate the more natural association of $T_{\tau}$ with time and $F_{\theta}$ with frequency. That is, the unitary time operator produces a translation in the natural time-representation (with respect to the eigensignals of $T$) of $s$, whereas the unitary frequency operator produces a translation in the frequency-representation of the signal. Note that the time and frequency variables are based on $(R_{+}, +)$, with the usual Fourier transform being the usual Fourier transform, that is $F_{ST} = IF_{ST}$, where $(F_{ST}) = \int s(t)e^{-j2\pi ft}dt$. Thus, the unitary operators for time and frequency can be interpreted as shift operators with shift being the group operation, simple translation, in this case. We now make the notion of shift precise for arbitrary groups.

Definition—Shift Operator: Let $k \in G$ be a variable associated with the Hermitian operator $K_{k} = S_{k}A\Sigma S_{k^\circ}$, and let $\lambda \in \Gamma$ be another variable associated with $L = S_{L}A\Sigma S_{L}$. Then, a (unitary) shift operator $K_{k}$, $k' \in G$, for the variable $k$, is one that produces a group translation in the natural $k$-representation of the signal, that is, $(S_{k}K_{k\circ}S_{k})(\lambda) = (S_{k^\circ}(k\circ k')^{-1})$. Similarly, a shift operator, $L_{k'}$, $\lambda' \in \Gamma$, for the variable $\lambda$, is one that satisfies $(S_{k}L_{k'}\Sigma)(\lambda) = (S_{\lambda}(k'\circ k)^{-1})$. Thus, based on the above arguments, the unitary operator associated with a variable should be the corresponding shift operator. We now characterize the shift operator for a given variable. In terms of the group translation operators $T_{k}$ and $T_{k'}$, which were defined in (3) and (5), the shift operators defined above can be expressed as

$$ K_{k'} = S_{k^\circ}T_{k'}^{-1}S_{k^\circ} \quad \text{and} \quad L_{\lambda'} = S_{L}T_{\lambda'}^{-1}S_{L}. $$

Moreover, time and frequency variables are dual in the sense of the definition of Section II-A since the eigensignals of $T$ and $F$ are related as $S_{F} = IF_{ST}^{-1}S_{T}$.

Using the notions of shift and dual operators, we show

27 The operator $K_{k}$ is a similar to the parameterized warping operator used in [10] to relate the power class to the affine class.

28 We note that the form of the affine WD in Baraniuk’s method can also be interpreted, due to the warping operations involved, in terms of a different parameterization of the affine group.

29 Moreover, time and frequency variables are dual in the sense of the definition of Section II-A since the eigensignals of $T$ and $F$ are related as $S_{F} = IF_{ST}^{-1}S_{T}$.
Substituting the fundamental relationships (7) in the above equations yields the following characterization of the shift operators in terms of their spectral representations:

\[ K_{k'} = S_{K'}^{-1} F_G^{-1} A_{G} S_K, \]

\[ S_K = F_G S_{K'}, \quad (66) \]

\[ L_{\lambda'} = S_{\lambda'}^{-1} F_G A_{G} S_{L}, \]

\[ S_L = F_G^{-1} S_{\lambda'}. \quad (67) \]

The uniqueness of shift operators follows from the uniqueness of the spectral representation of groups of unitary operators [21].

**Discussion:** By Stone’s theorem, the most direct choice of unitary operators to be associated with the variables \( k \) and \( \lambda \) is the character operator \( K_k \equiv (K_k, \nu) \) and \( L_{\lambda} \equiv (I, L) \), respectively (which corresponds to associating time with \( F_{\theta} \) and frequency with \( T_{\nu} \), in the case \( G = \mathbb{R} \)). However, we have argued that if we start with a variable and a corresponding Hermitian operator, the appropriate unitary operator to be associated with the variable is the shift operator, which is precisely the dual of the unitary (character) operator defined by the Hermitian operator via Stone’s theorem. This choice for unitary operator correspondence is further justified by the fact that given a variable belonging to a group, the parameter of the corresponding shift operator belongs to the group itself, whereas the parameter of the character operator belongs to the dual group.

**B. CED and IED Marginals**

Consider two variables \( k \) and \( \lambda \) with the corresponding Hermitian operators \( K_k \) and \( L_{\lambda} \), as defined in the previous section. Suppose we generate joint \( k, \lambda \) distributions using the Hermitian operator counterpart of Baraniuk’s method outlined in Section III. If we choose the shift operators \( K_k' \) and \( L_{\lambda'} \), as the unitary operators corresponding to \( k \) and \( \lambda \), respectively, then by definition of shift operators, we note from (30) and (31) that the \( k \) marginal is \( K \)-covariant and \( \lambda \) marginal is \( L \)-covariant. On the other hand, if we choose the character operators \( K_k \) and \( L_{\lambda} \) as the corresponding unitary operators, then the marginals are invariant to the unitary operators. Thus, we see that the Hermitian operator-based approach can generate both CED and IED marginals; it is simply a matter of associating the appropriate unitary operators with the variables. In fact, in light of our argument for associating shift operators with variables, the Hermitian operator approach always generates CED marginals.

**VI. CONCLUSION**

Recently, in an attempt to tailor JSR’s to a wide variety of signal characteristics, there has been significant research on joint distributions of variables other than time and frequency. Time-scale representations constituted the first such generalizations spurred by the interest in the wavelet transform. More recently, research in such generalized JSR’s has culminated in two main classes of general theories for JSR’s of arbitrary variables. On one hand are the distributional approaches of Cohen [1], [6] and Baraniuk [16], [17], which emphasize the marginal properties of JSR’s and, on the other hand, are the covariance-based formulations proposed by Hlawatsch and Böckike [18], [19] and Sayeed and Jones [20], which are based on the covariance of JSR’s to certain unitary operators.

In this paper, we focussed on the general methods proposed by Cohen and Baraniuk, and one of the main results of the paper was that the two approaches, despite being apparently quite different, are completely equivalent. In addition, we explicitly characterize the relationship between the operators associated with the variables of interest in the two methods. From a theoretical viewpoint, by unifying the two main distributional approaches, the results of this paper facilitate a better understanding of the theory of JSR’s of arbitrary variables.

Quite remarkably, the two types of JSR’s generated by the two methods are *unitarily equivalent*, and the unitary transformations relating them are simply input- and output-axis warping transformations. This fact has important implications for properties of arbitrary JSR’s. In particular, by appropriately prewarping the signal axis and postwarping the axes of JSR’s from Cohen’s method (which possess translational covariance properties), we can generate JSR’s with radically different covariance properties. These warped representations (the JSR’s in Baraniuk’s method), in view of our equivalence results, can be interpreted as JSR’s corresponding to the *same* variables, albeit with respect to different bases or signal spaces. Such flexibility in choosing covariance properties can potentially be very useful for detecting or estimating the effects of certain unitary signal transformations (which may model a channel or system, for example) [34], [35]; from a practical perspective, certain changes are easier than others to detect or estimate.

Fundamental to both Cohen’s and Baraniuk’s approach is the idea of associating variables with operators; Cohen uses Hermitian operators, whereas Baraniuk uses groups of unitary operators. An important issue, which is fundamental to the understanding of generalized JSR’s and which has been left unaddressed in existing treatments, is the relationship between the two types of operator correspondences. By interpreting Baraniuk’s approach in terms of Hermitian operators and using the concept of shift and dual operators, we have precisely characterized the relationship between the Hermitian and unitary operators associated with a variable. In particular, given a variable and a Hermitian operator associated with it, the corresponding unitary operator should be the shift operator associated with the variable.

Finally, we note that Baraniuk’s method may conceptually be the preferred method in certain situations, but Cohen’s approach, being based on the Fourier transform, has a computational advantage. In fact, an important practical implication of the equivalence results is that we can replace the group transforms in Baraniuk’s approach, which are computationally inefficient in general, with the Fourier transform and simple pre- and post-processing. Moreover, depending on the variable in question, either the corresponding Hermitian or the unitary

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31 The *character* and *shift* operators are dual (see Section II-A) because

\[ S_{K'} = S_{K'}^* = F_{\phi} S_K, \]

and

\[ S_L = S_L^* = F_{\phi} S_L, \]

where we have used the fact that a Hermitian operator and the corresponding character operator share the same eigenfunctions (e.g., \( S_{K'} = S_{K'}^* \)).
operator may have a simpler description. Thus, the results of this paper, in addition to providing a better understanding of the theory and application of generalized JSR’s, also allow the adoption of the most convenient approach in any given situation.

APPENDIX
PROOF OF THEOREM 1

We first need to verify that \( \varphi \) defined in (33) is one-to-one and onto \( \Gamma_2 \) and satisfies

\[
\varphi(\kappa_1 \circ \kappa_2) = \varphi(\kappa_1) \cdot \varphi(\kappa_2) \quad \text{for all } \kappa_1, \kappa_2 \in \Gamma_1. \tag{68}
\]

To show \( \varphi \) is one-to-one, suppose that \( \varphi(\kappa_1) = \varphi(\kappa_2) \). Then, from (33), we have

\[
(\psi^{-1}(l), \kappa_1)_{G_2} = (l, \varphi(\kappa_1))_{G_2} = (l, \varphi(\kappa_2))_{G_2} = (\psi^{-1}(l), \kappa_2)_{G_1}, \quad \text{for all } l \in G_2 \tag{69}
\]

which implies, using the fact that \( \psi \) is onto, that \( (k, \kappa_1 \circ k^{-1}) = 1 \) for all \( k \in G_1 \), from which it follows that \( \kappa_1 = \kappa_2 \) [22], which proves that \( \varphi \) is one-to-one.

For ontoeness, we need to show that for each \( \lambda \in \Gamma_2 \), there exists a \( \kappa \in \Gamma_1 \) such that \( \varphi(\kappa) = \lambda \). Given \( \lambda \in \Gamma_2 \), define \( \kappa \in \Gamma_1 \) as

\[
(\kappa, \lambda)_{G_2} \equiv (\varphi(\kappa), \lambda)_{G_2}, \quad \text{for all } k \in G_1. \tag{70}
\]

Then, by (33) we have, for all \( l \in G_2 \),

\[
(l, \varphi(\kappa_1 \circ \kappa_2))_{G_2} = (\psi^{-1}(l), \kappa_1 \circ \kappa_2)_{G_1} = (\psi^{-1}(l), \kappa_1)_{G_1} \cdot (\psi^{-1}(l), \kappa_2)_{G_1}, \tag{71}
\]

by definition of characters \( \in \Gamma_1 \)

\[
= (l, \varphi(\kappa_1))_{G_2} \cdot (l, \varphi(\kappa_2))_{G_2} \quad \text{by definition of characters } \in \Gamma_2 \tag{72}
\]

\[
= (l, \varphi(\kappa_1) \cdot \varphi(\kappa_2))_{G_2} \quad \text{by definition of characters } \in \Gamma_2 \tag{73}
\]

from which we conclude that (68) holds.

We now prove (34). We first note that the set function \( \mu_{\Gamma_1} \cdot \Gamma_1 \rightarrow [0, \infty) \) defined by \( \mu_{\Gamma_1}(F) = \mu_{\Gamma_2}(\varphi(F)) \) for all measurable sets \( F \subset \Gamma_1 \) is also a Haar measure on \( \Gamma_1 \), and thus, by the uniqueness of Haar measure [22], we must have \( \mu_{\Gamma_1} = c \mu_{\Gamma_1} \) for some \( c > 0 \). Thus, we only need to show that \( c = 1 \), which can be shown by proving (34) for any one particular measurable set \( F \subset \Gamma_1 \). Let \( F \subset \Gamma_1 \) be a measurable set with \( \mu_{\Gamma_1}(F) < \infty \). Define \( H \in L^2(\Gamma_1, d\mu_{\Gamma_1}) \) by \( H(\kappa) \equiv I_F(\kappa) \), which is the indicator function of \( F \). Then,

\[
h = F^{-1}_G H \text{ exists in } L^2(G_1, d\mu_{G_1}) \text{ and using } (1), \text{ we have}
\]

\[
H(\kappa) = \int_{G_1} h(k)(k, \kappa)_{G_1} d\mu_{G_1}(k)
\]

\[
= \int_{G_1} (h(k)(\psi(k), \varphi(\kappa))_{G_2} d\mu_{G_2}(k)
\]

using (33),

\[
H(\varphi^{-1}(\lambda)) = \int_{G_2} h(\psi^{-1}(l))(l, \lambda)_{G_2} d\mu_{G_2}(\psi^{-1}(l)),
\]

\[
l = \psi(k), \lambda = \varphi(\kappa) \tag{76}
\]

using (32).

By inverting (77), we get

\[
h(\psi^{-1}(l)) = \int_{\Gamma_2} H(\varphi^{-1}(\lambda))(l, \lambda)_{G_2} d\mu_{\Gamma_2}(\lambda). \tag{78}
\]

Note that \( \psi^{-1}(0) = 0 \) and that the relation \( h = F^{-1}_GH \) implies that \( h(0) = \int_{\Gamma_1} H d\mu_{\Gamma_1} = \mu_{\Gamma_1}(F) \). Thus, from (78), we have

\[
\mu_{\Gamma_1}(F) = h(0) = h(\psi^{-1}(0)) = \int_{\Gamma_2} I_F(\psi^{-1}(\lambda)) d\mu_{\Gamma_2}(\lambda)
\]

\[
= \int_{\Gamma_2} I_{\varphi(F)}(\lambda) d\mu_{\Gamma_2}(\lambda) = \mu_{\Gamma_2}(\varphi(F)) \tag{79}
\]

which proves that \( c = 1 \) and, hence, (34). This completes the proof.

REFERENCES


Akbar M. Sayeed (S’89) received the B.S. degree from the University of Wisconsin-Madison in 1991 and the M.S. and Ph.D. degrees in 1993 and 1996, respectively, from the University of Illinois at Urbana-Champaign, all in electrical engineering. While at the University of Illinois, he was a research assistant with the Coordinated Science Laboratory of the University of Illinois. From 1992 to 1995, he was also the Schlumberger Fellow in signal processing. Currently, he is a postdoctoral fellow at Rice University, Houston, TX. His research interests are in statistical and nonstationary signal processing, time-frequency and wavelet analysis, and signal processing for wireless communications.

Douglas L. Jones (M’87) received the B.S.E.E., M.S.E.E., and Ph.D. degrees from Rice University, Houston, TX, in 1983, 1986, and 1987, respectively. During the 1987-1988 academic year, he was at the University of Erlangen-Nuremberg in Germany on a Fulbright postdoctoral fellowship. Since 1988, he has been with the University of Illinois at Urbana-Champaign, where he is currently an Associate Professor. He is an author of the laboratory textbook A Digital Signal Processing Laboratory Using the TMS32010. His research interests are in digital signal processing, including time-frequency and time-varying signal analysis, efficient algorithms for VLSI implementation, and various applications.