IV. Discussion and Conclusions

The GC estimate has been shown to be invariant with respect to the statistical behavior of $\boldsymbol{x}_1$, provided that $\boldsymbol{x}_2, \cdots, \boldsymbol{x}_M$ have stationary Gaussian distributions and are statistically independent of $\boldsymbol{x}_1$. This invariance extends the utility of the GC estimate from passive to active detection scenarios. An example simulating a three-channel matched filter scenario was described. This example demonstrated that a GC-based multiple-channel matched filter can, at least in some cases, provide better detection performance than is obtained using multiple individual two-channel MSC-based detectors.

Other applications, in the area of cyclostationary signal detection as suggested by [8], for example, appear promising but need further investigation.

References

[1] G. C. Carter, C. Knapp, and A. H. Nuttall, "Estimation of the magnitude-squared coherence function via overlapped fast Fourier transform processing," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 331–344, Aug. 1973.

[2] H. Gish and D. Cochran, "Generalized coherence," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Apr. 1987, vol. 5, pp. 2745–2748.

[3] D. Cochran, H. Gish, and D. Sinno, "A geometric approach to multiple-channel signal detection," *IEEE Trans. Signal Processing*, vol. 43, pp. 2049–2057, Sept. 1995.

[4] A. H. Nuttall, "Invariance of distribution of coherence estimate to second-channel statistics," *IEEE Trans. Acoust, Speech, Signal Processing*, vol. ASSP-29, pp. 120–122, Feb. 1981.

[5] H. Gish and D. Cochran, "Invariance of the magnitude-squared coherence estimate with respect to second-channel statistics," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1774–1776, Dec. 1987.

[6] D. Sinno and D. Cochran, "Invariance of the generalized coherence estimate with respect to reference channel statistics," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Mar. 1992, vol. 2, pp. 505–508.

[7] P. J. Davis, *Interpolation and Approximation*. New York: Dover, 1975.

[8] S. Enserink and D. Cochran, "On detection of cyclostationary signals," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, May 1995, vol. 3, pp. 2004–2007.

On the Equivalence of the Operator and Kernel Methods for Joint Distributions of Arbitrary Variables

Akbar M. Sayeed

***Abstract*—Generalizing the concept of time-frequency representations, Cohen has recently proposed a method, based on operator correspondence rules, for generating joint distributions of arbitrary variables. As an alternative to considering all such rules, which is a practical impossibility in general, Cohen has proposed the kernel method in which different distributions are generated from a fixed rule via an arbitrary kernel. In this correspondence, we derive a simple but rather stringent necessary condition, on the underlying operators, for the kernel method (with the kernel functionally independent of the variables) to generate *all* bilinear distributions. Of the specific pairs of variables that have been studied, essentially only time and frequency satisfy the condition; in particular, the important variables of time and scale do not. The results warrant further study for a systematic characterization of bilinear distributions in Cohen's method.**

I. Introduction

Time-frequency representations (TFR's), such as the Wigner distribution and the short-time Fourier transform, represent signal characteristics jointly in terms of time and frequency and are powerful tools for nonstationary signal analysis and processing [1]. However, due to their inherent structure, TFR's can accurately represent only a limited class of nonstationary signal characteristics. In an effort to expand the applicability of joint signal representations to a broader class of signals, substantial amount of research has been directed to the study of joint distributions of variables other than time and frequency [2]–[7]. Spurred by the interest in the wavelet transform [8], joint time-scale representations constituted the first such generalizations [2], [3] and have received considerable attention.

In view of this recent trend, general theories for joint distributions of arbitrary variables have been proposed by many authors [1], [5], [9]–[11]. The first such generalization was proposed by Scully and Cohen [12] and developed by Cohen [1], [5] in direct extension of his original method for generating joint TFR's [13]. Baraniuk proposed a general approach based on group theoretic arguments [9] that was shown by Sayeed and Jones [14], [15] to be equivalent to Scully and Cohen's method. Other covariance-based generalizations have also been proposed [10], [11], [16], which complement Cohen's distributional method by characterizing joint representations in terms of covariance properties. However, Cohen's method seems to be the most general approach to date since no joint group structure is imposed on the variables as is done in [10], [11], and [16].

Fundamental to Cohen's method is the idea of associating variables with Hermitian (self-adjoint) operators [1]. For given variables, the entire class of joint distributions is generated by the infinitely many (in general) operator correspondence rules for an exponential function of the variables (the characteristic function *operator method* [1]). As an alternative to considering all possible correspondence rules, which is a practical impossibility in general, Cohen has proposed the *kernel method* in which a fixed operator correspondence is used and different

Manuscript received November 15, 1995; revised May 27, 1996. This work was supported by the Joint Services Electronics Program under Grant N00014-90-J-1270. The associate editor coordinating the review of this paper and approving it for publication was Dr. Nurgun Erdol.

The author was with the Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801 USA. He is now with the Department of Electrical and Computer Engineering, Rice University, Houston, TX 77005 USA.

Publisher Item Identifier S 1053-587X(97)02584-1.

joint distributions are generated via an arbitrary kernel. In [17], a simple counterexample is constructed to show that the kernel method does not generate all the correspondence rules for the variables of time and scale.

In this correspondence, we show that for two variables, say, a and b, the corresponding Hermitian operators $\mathcal{A}$ and $\mathcal{B}$ must satisfy rather stringent conditions, such as

$$e^{j\alpha\mathcal{A}}e^{j\beta\mathcal{B}} = e^{jf(\alpha,\beta)}e^{j\beta\mathcal{B}}e^{j\alpha\mathcal{A}}, \text{ for all } (\alpha,\beta) \in \mathbb{R}^2, \text{ and for some } f\colon \mathbb{R}^2 \to \mathbb{R} \tag{1}$$

for the kernel method[1] to generate the entire class of bilinear distributions determined by the operator method. We also generalize the result to an arbitrary number of variables and argue that of the specific variables considered in the literature, only time and frequency (and those unitarily equivalent to time and frequency) satisfy the conditions. We begin with a brief description of Cohen's method.

II. Cohen's Method

We describe the method for two variables; extension to more variables will be obvious. We assume that all signals of interest belong to $L^2(\mathbb{R})$, which is the space of finite energy signals.

The characteristic function M of a joint a-b distribution P of signal s is defined as [1]

$$(Ms)(\alpha,\beta) = \iint (Ps)(a,b)e^{j2\pi\alpha a}e^{j2\pi\beta b}\, da\, db \tag{2}$$

and the distribution can be recovered from M as

$$(Ps)(a,b) = \iint (Ms)(\alpha,\beta)e^{-j2\pi\alpha a}e^{-j2\pi\beta b}\, d\alpha\, d\beta. \tag{3}$$

The key observation is that the characteristic function can be directly computed from the signal by using a characteristic function operator $\boldsymbol{M}^{(\alpha,beta)}$ corresponding to the function $e^{j2\pi\alpha a}e^{j2\pi\beta b}$, as

$$(Ms)(\alpha,\beta) = \langle \boldsymbol{M}^{(\alpha,\beta)}s, s\rangle \equiv \int (\boldsymbol{M}^{(\alpha,\beta)}s)(x)s^*(x)\, dx. \tag{4}$$

Since the operators $\mathcal{A}$ and $\mathcal{B}$ do not commute in general, there are infinitely many ways in which the function $e^{j2\pi\alpha a}e^{j2\pi\beta b}$ can be associated with an operator; three prominent examples are $e^{j2\pi(\alpha\mathcal{A}+\beta\mathcal{B})}$ (Weyl correspondence), $e^{j2\pi\alpha\mathcal{A}}e^{j2\pi\beta\mathcal{B}}$, and $e^{j2\pi\beta\mathcal{B}}e^{j2\pi\alpha\mathcal{A}}$, which we will use throughout this correspondence. The corresponding infinitely many joint distributions can then be recovered via (3), and they *define* the entire class of joint a-b distributions.

In order to characterize all the different correspondence rules and, hence, the entire class of joint a-b distributions, Cohen has proposed the kernel method, which assumes that all characteristic functions can be generated by weighting any one particular one with an arbitrary kernel [1, p. 229]. That is, given a particular characteristic function, say M_o, *all* the infinitely many characteristic functions can be generated as

$$(M(\phi))(s)(\alpha,\beta) = (M_o s)(\alpha,\beta)\phi(\alpha,\beta) \tag{5}$$

[1] We restrict the discussion to kernels that are functionally independent of the signal and the variables. See remarks in footnote 3 on the issue of kernel dependence.

where ϕ is the weighting kernel.[2] The corresponding joint distributions $P(\phi)$ can then be recovered by using (5) in (3). In the case of time-frequency, fixing M_o to be the Weyl correspondence yields the following commonly used characterization of Cohen's class of TFR's first proposed in [13]

$$(C(\phi)s)(t,f) = \iiint \phi(\theta,\tau)s(u+\tau/2) \cdot s^*(u-\tau/2)e^{j2\pi\theta(u-t)}e^{-j2\pi\tau f}\, du\, d\theta\, d\tau. \tag{6}$$

III. Necessary Conditions for the Validity of the Kernel Method

According to Cohen's kernel method, any two characteristic functions, say, M_1 and M_2, corresponding to two different operator correspondences $\boldsymbol{M}_1^{(\alpha,\beta)}$ and $\boldsymbol{M}_2^{(\alpha,\beta)}$ must be related by

$$(M_1 s)(\alpha,\beta) \equiv \langle \boldsymbol{M}_1^{(\alpha,\beta)}s, s\rangle = \phi(\alpha,\beta)(M_2 s)(\alpha,\beta) \equiv \phi(\alpha,\beta)\langle \boldsymbol{M}_2^{(\alpha,\beta)}s, s\rangle, \quad \text{for all } s \in L^2(\mathbb{R}) \tag{7}$$

for some $\phi :\ \mathbb{R}^2 \to \mathbb{C}$. It follows that a necessary and sufficient condition for the kernel method to hold is that *any* two operator correspondences must be related by[3]

$$\boldsymbol{M}_1^{(\alpha,\beta)} = \phi(\alpha,\beta)\boldsymbol{M}_2^{(\alpha,\beta)} \tag{8}$$

for some ϕ. In particular, the above relationship must hold in the case when both $\boldsymbol{M}_1^{(\alpha,\beta)}$ and $\boldsymbol{M}_2^{(\alpha,\beta)}$ are unitary operators,[4] in which case, it can be easily verified that $|\phi(\alpha,\beta)| = 1$ for all (α,β). For example, all characteristic function operators of the following form are unitary[5]

$$\boldsymbol{M}^{(\alpha,\beta)} = \prod_k e^{j2\pi\gamma_k\mathcal{C}_k} \qquad \text{where} \tag{9}$$

$$\mathcal{C}_k = \mathcal{A} \text{ or } \mathcal{B}, \quad \gamma_k = \begin{cases} \alpha_k & \text{if } \mathcal{C}_k = \mathcal{A} \\ \beta_k & \text{if } \mathcal{C}_k = \mathcal{B} \end{cases}, \quad \text{and} \quad \sum_k \alpha_k = \alpha, \quad \sum_k \beta_k = \beta. \tag{10}$$

Two specific cases are the correspondences

$$\boldsymbol{M}_1^{(\alpha,\beta)} = e^{j2\pi\alpha\mathcal{A}}e^{j2\pi\beta\mathcal{B}} \quad \text{and} \quad \boldsymbol{M}_2^{(\alpha,\beta)} = e^{j2\pi\beta\mathcal{B}}e^{j2\pi\alpha\mathcal{A}}$$

which result in the relationship (1). Extension to more than two variables immediately follows, and we have the following general result.

Proposition: Let $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_N$ be the Hermitian operators corresponding to the N variables $a_1, a_2, \cdots, a_N$ in Cohen's method. Then, a necessary condition for the kernel method to generate *all* bilinear joint distributions of $a_1, a_2, \cdots, a_N$ is that for *any* two *unitary* characteristic function operator correspondences $\boldsymbol{M}_1^{(\alpha_1,\alpha_2,\cdots,\alpha_N)}$

[2] Cohen does not preclude the possibility of functional dependence of the kernel on the variables and the signal [1, p. 140]. However, we restrict the discussion to the important case of bilinear distributions, which precludes signal-dependent kernels. Moreover, we are interested in a characterization of bilinear distributions in which the kernel is not a function of the variables, as is true for all covariance-based generalizations [10], [11], and for Cohen's class of bilinear TFR's [1] and the affine class of bilinear time-scale representations [3], [2], in particular.

[3] We use the fact that if $\boldsymbol{A}$ is a linear operator on a complex inner product space $\mathcal{H}$, then $\langle \boldsymbol{A}s, s\rangle = 0$ for all $s \in \mathcal{H} \Leftrightarrow \boldsymbol{A} \equiv \boldsymbol{0}$; see, for example, [18, p. 374].

[4] An operator $\boldsymbol{U}$ is unitary if $\langle \boldsymbol{U}s, \boldsymbol{U}s\rangle = \langle s, s\rangle$ for all s.

[5] Which follows from the fact that $e^{j\mathcal{A}}$ is a unitary operator if $\mathcal{A}$ is Hermitian and that the composition of unitary operators is unitary.

and $\boldsymbol{M}_2^{(\alpha_1,\alpha_2,\cdots,\alpha_N)}$, the following relationship must hold for all $(\alpha_1,\alpha_2,\cdots,\alpha_N) \in \mathbb{R}^N$:

$$\boldsymbol{M}_1^{(\alpha_1,\alpha_2,\cdots,\alpha_N)} = e^{jf(\alpha_1,\alpha_2,\cdots,\alpha_N)}\boldsymbol{M}_2^{(\alpha_1,\alpha_2,\cdots,\alpha_N)} \quad \text{for some } f\colon \mathbb{R}^N \to \mathbb{R}. \tag{11}$$

Corollary: A particular necessary condition for the validity of the kernel method is

$$e^{j\mathcal{A}_1}e^{j\mathcal{A}_2}\cdots e^{j\mathcal{A}_N} = e^{jc}e^{j\mathcal{A}_N}e^{j\mathcal{A}_{N-1}}\cdots e^{j\mathcal{A}_1}, \quad \text{for some } c \in \mathbb{R}. \tag{12}$$

IV. Examples

1) Time and Frequency: Defining the time and frequency Hermitian operators as

$$(\mathcal{T}s)(t) = ts(t) \quad \text{and} \quad (\mathcal{F}s)(t) = -\frac{j}{2\pi}\dot{s}(t)$$

[1][6], respectively, we have

$$(e^{j2\pi\theta\mathcal{T}}s)(t) = e^{j2\pi\theta t}s(t) \quad \text{and} \quad (e^{j2\pi\tau\mathcal{F}}s)(t) = s(t+\tau)$$

[1]. The following relationships hold between the three main correspondences [1, p. 155]

$$e^{j2\pi(\theta\mathcal{T}+\tau\mathcal{F})} = e^{-j\pi\theta\tau}e^{j2\pi\tau\mathcal{F}}e^{j2\pi\theta\mathcal{T}} = e^{j\pi\theta\tau}e^{j2\pi\theta\mathcal{T}}e^{j2\pi\tau\mathcal{F}} \tag{13}$$

from which it can be easily verified that all the three correspondences satisfy (11) pairwise. In fact, the relationships (13) can be used to show that the necessary and sufficient condition (8) is satisfied for all pairs of orderings, and thus, the kernel-based characterization (6) does indeed generate all possible bilinear joint time-frequency distributions.

2) Time and Scale: Define the operator $\mathcal{C} = \frac{1}{2}(\mathcal{T}\mathcal{F}+\mathcal{F}\mathcal{T})$ that is associated with scale in [1].[7] The corresponding exponential operator is the scaling operator $(e^{j2\pi\sigma\mathcal{C}}s)(t) = e^{\sigma/2}s(e^{\sigma}t)$ [1]. The three main characteristic function operators for $\mathcal{T}$ and $\mathcal{C}$ are related as [1]

$$\begin{aligned} e^{j2\pi(\theta\mathcal{T}+\sigma\mathcal{C})} &= \exp\left[j2\pi\theta\left(\frac{e^{\sigma}-1}{\sigma} - e^{\sigma}\right)\mathcal{T}\right]e^{j2\pi\sigma\mathcal{C}}e^{j2\pi\theta\mathcal{T}} \\ &= \exp\left[j2\pi\theta\left(\frac{e^{\sigma}-\sigma-1}{\sigma}\right)\mathcal{T}\right]e^{j2\pi\theta\mathcal{T}}e^{j2\pi\sigma\mathcal{C}}, \quad \text{or} \end{aligned} \tag{14}$$

$$\begin{aligned} e^{j2\pi(\theta\mathcal{T}+\sigma\mathcal{C})} &= e^{j2\pi\sigma\mathcal{C}}e^{j2\pi\theta\mathcal{T}}\exp\left[j2\pi\theta\left(\frac{1-e^{-\sigma}}{\sigma} - e^{-2\sigma}\right)\mathcal{T}\right] \\ &= e^{j2\pi\theta\mathcal{T}}e^{j2\pi\sigma\mathcal{C}} \\ &\quad \cdot \exp\left[j2\pi\theta\left(\frac{(1-e^{-\sigma})(\sigma+1)}{\sigma} - e^{-2\sigma}\right)\mathcal{T}\right], \quad \text{and} \end{aligned} \tag{15}$$

$$\begin{aligned} e^{j2\pi\theta\mathcal{T}}e^{j2\pi\sigma\mathcal{C}} &= e^{j2\pi\sigma\mathcal{C}}e^{j2\pi\theta\mathcal{T}}e^{j2\pi\theta(e^{-\sigma}-1)\mathcal{T}} \\ &= e^{j2\pi\theta(1-e^{\sigma})\mathcal{T}}e^{j2\pi\sigma\mathcal{C}}e^{j2\pi\theta\mathcal{T}}. \end{aligned} \tag{16}$$

We note that none of the unitary characteristic function operators is simply a scalar multiple of the others for arbitrary values of the parameters; instead of a weighting function, an *operator* relates pairs of correspondences. Thus, the condition of the Proposition [and, in particular, (1)] is violated, and hence, the kernel method does not generate all joint $\mathcal{T}$-$\mathcal{C}$ distributions. Indeed, a specific counterexample is constructed in [17] to show that the characteristic functions corresponding to the two correspondences in (16) are not related by a weighting kernel.

Similarly, it can be easily verified by using the Corollary to the Proposition that the joint frequency-scale and time-frequency-scale distributions discussed in [1, p. 258–259] are not completely characterized by the kernel method.

V. Discussion

The necessary condition stated in the Proposition is rather stringent. To appreciate this, we use the Baker–Campbell–Hausdorff formula, which can be stated, to third order, as [20]

$$e^{j\mathcal{A}}e^{j\mathcal{B}} = \exp\left[j\left(\mathcal{A}+\mathcal{B}+\frac{j}{2}\mathcal{D} - \frac{1}{12}[\mathcal{A},\mathcal{D}] - \frac{1}{12}[\mathcal{D},\mathcal{B}]\cdots\right)\right] \tag{17}$$

where $\mathcal{D} = [\mathcal{A},\mathcal{B}] \equiv \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is the commutator operator. One of the simplest nontrivial special cases is when the commutator commutes with both the operators, in which case, we have the following relationships [1], [20]:

$$\begin{aligned} e^{j\mathcal{A}}e^{j\mathcal{B}} &= e^{j(\mathcal{A}+\mathcal{B})}e^{-(1/2)\mathcal{D}} = e^{-(1/2)\mathcal{D}}e^{j(\mathcal{A}+\mathcal{B})} \\ &= e^{j\mathcal{B}}e^{j\mathcal{A}}e^{-\mathcal{D}} = e^{-\mathcal{D}}e^{j\mathcal{B}}e^{j\mathcal{A}}. \end{aligned} \tag{18}$$

Even in this case, unless $\mathcal{D} = jc\boldsymbol{I}, c \in \mathbb{R}$, the conditions of the Proposition are violated, and the kernel method does not hold. Note that for time and frequency, $\mathcal{D} = (j/2\pi)\boldsymbol{I}$, and (18) yields (13), making the kernel method work. Moreover, since this commutator relationship does not change for operators that are unitarily equivalent to time and frequency [21], [19], joint distributions of variables that are unitarily equivalent to time and frequency [7], [21], [19] are also completely characterized by the kernel method.

Another special case studied in [1] regarding joint distributions involving scale is

$$\mathcal{D} = [\mathcal{A},\mathcal{B}] = c_1\boldsymbol{I} + c_2\mathcal{A} \tag{19}$$

which results in the relationship [1, p. 228]

$$e^{j\alpha\mathcal{A}+j\beta\mathcal{B}} = e^{j\mu\alpha c_1/c_2}e^{j\alpha\mu\mathcal{A}}e^{j\beta\mathcal{B}}e^{j\alpha\mathcal{A}} \tag{20}$$

where $c_1, c_2 \in \mathbb{C}$, and

$$\mu = \frac{1}{j\beta c_2}[1-(1+j\beta c_2)e^{-j\beta c_2})].$$

Again, (20) implies that the condition (11) is violated for the correspondences

$$\boldsymbol{M}_1^{(\alpha,\beta)} = e^{j\alpha\mathcal{A}+j\beta\mathcal{B}} \quad \text{and} \quad \boldsymbol{M}_2^{(\alpha,\beta)} = e^{j\beta\mathcal{B}}e^{j\alpha\mathcal{A}}$$

and as a specific example of this case, we showed in the last section that the kernel method does not hold for joint $\mathcal{T}$-$\mathcal{C}$ distributions (and, thus, for variables that are unitarily equivalent to time and scale).[8]

[6] Cohen uses the radian frequency operator $(\mathcal{W}s)(t) = -j\dot{s}(t)$.

[7] A different correspondence for scale is argued in [7] and [19]. However, even for that correspondence, the kernel method does not hold for time and scale.

[8] It can be readily verified that of the specific pairs of variables that have been studied in the literature (see, for example, [1], [4], [19], [7]), only time and frequency and those unitarily equivalent to time and frequency satisfy (12). Moreover, the above discussion based on the BCH formula suggests that it may even be true, in general, that no other pairs satisfy the condition.

VI. Conclusions

Cohen's general method for generating distributions of arbitrary variables, when viewed from the perspective of operator correspondences, is a powerful and versatile tool. However, characterizing all the different operator correspondences is nontrivial, and the simple kernel method proposed by Cohen does not encompass all possible correspondences in general. In fact, the necessary conditions derived in this paper for the validity of the kernel method are rather stringent and, for pairs of variables that have been studied in the literature, we argue that they hold only for time and frequency and for variables that are unitarily equivalent to time and frequency.

Thus, in general, applying the kernel method to a particular correspondence rule generates a proper subset of the entire class of joint distributions. However, it is conceivable that the families of joint distributions generated by a finite set of correspondence rules, via the kernel method, may cover the entire class of joint distributions.

It is worth noting that covariance-based generalizations of joint distributions [10], [11], [16], which necessarily impose a joint group structure on the variables, naturally yield a kernel method that generates all the joint distributions in the class. Thus, it might be fruitful to study the relationship between the two approaches for arbitrary joint distributions in order to develop a systematic characterization of all the correspondence rules in Cohen's general method.

Acknowledgment

The author would like to thank R. Baraniuk and L. Cohen for insightful discussions.

References

[1] L. Cohen, *Time-Frequency Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[2] J. Bertrand and P. Bertrand, "A class of affine Wigner distributions with extended covariance properties," *J. Math. Phys.*, vol. 33, no. 7, pp. 2515–2527, 1992.
[3] O. Rioul and P. Flandrin, "Time-scale distributions: A general class extending the wavelet transform," *IEEE Trans. Signal Processing*, vol. 40, pp. 1746–1757, May 1992.
[4] A. Papandreou, F. Hlawatsch, and G. F. Boudreaux-Bartels, "The hyperbolic class of time-frequency representations. Part I: Constant-Q warping, the hyperbolic paradigm, properties, and members," *IEEE Trans. Signal Processing*, vol. 41, pp. 3425–3444, Dec. 1993.
[5] L. Cohen, "A general approach for obtaining joint representations in signal analysis and an application to scale," in *Proc. SPIE 1566*, San Diego, July 1991.
[6] R. G. Shenoy and T. W. Parks, "Wide-band ambiguity functions and affine Wigner distributions," *Signal Processing*, vol. 41, no. 3, pp. 339–363, 1995.
[7] R. G. Baraniuk and D. L. Jones, "Unitary equivalence: A new twist on signal processing," *IEEE Trans. Signal Processing*, vol. 43, pp. 2269–2282, Oct. 1995.
[8] O. Rioul and M. Vetterli, "Wavelets and signal processing," *IEEE Signal Processing Mag.*, Oct. 1991.
[9] R. G. Baraniuk, "Beyond time-frequency analysis: energy densities in one and many dimensions," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Processing—ICASSP '94*, 1994, vol. 3, pp. 357–360.
[10] F. Hlawatsch and H. Bölcskei, "Displacement-covariant time-frequency energy distributions," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Processing—ICASSP '95*, 1995, pp. 1025–1028.
[11] A. M. Sayeed and D. L. Jones, "A canonical covariance-based method for generalized joint signal representations," *IEEE Signal Processing Lett.*, vol. 3, pp. 121–123, Apr. 1996.
[12] M. Scully and L. Cohen, "Quasiprobability distributions for arbitrary operators," in *The Physics of Phase Space*, Y. S. Kim and W. W. Zachary, Eds. New York: Springer Verlag, 1987.
[13] L. Cohen, "Generalized phase-space distribution functions," *J. Math. Phys.*, vol. 7, pp. 781–786, 1966.
[14] A. M. Sayeed and D. L. Jones, "On the equivalence of generalized joint signal representations," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Processing—ICASSP '95*, 1995, pp. 1533–1536.
[15] ——, "Equivalence of generalized joint signal representations of arbitrary variables," *IEEE Trans. Signal Processing*, vol. 44, pp. 2959–2970, Dec. 1996.
[16] R. G. Baraniuk, "Covariant time-frequency representations through unitary equivalence," *IEEE Signal Processing Lett.*, pp. 79–81, Mar. 1996.
[17] ——, "A limitation of the kernel method for joint distributions of arbitrary variables," *IEEE Signal Processing Lett.*, pp. 51–53, Feb. 1996.
[18] A. W. Naylor and G. R. Sell, *Linear Operator Theory in Engineering and Science*. New York: Springer-Verlag, 1982.
[19] A. M. Sayeed and D. L. Jones, "Integral transforms covariant to unitary operators and their implications for joint signal representations," *IEEE Trans. Signal Processing*, vol. 44, pp. 1365–1377, June 1996.
[20] R. M. Wilcox, "Exponential operators and parameter differentiation in quantum physics," *J. Math. Phys.*, vol. 8, pp. 962–982, 1967.
[21] R. G. Baraniuk and L. Cohen, "On joint distributions of arbitrary variables," *IEEE Signal Processing Lett.*, vol. 2, pp. 10–12, Jan. 1995.

Block Sampling Rate Conversion Systems Using Real-Valued Fast Cyclic Convolution Algorithms

Hideo Murakami

***Abstract*—The recently obtained real-valued transform is related to the polyphase decomposition of a sequence. This observation is applied for deriving sampling rate conversion systems that are implemented by the real-valued fast cyclic convolution algorithms. The systems include interpolation by an integer factor, decimation by an integer factor, and sampling rate conversion by a rational factor. The proposed implementations are useful when signals and impulse responses of filters are restricted to be real.**

I. Introduction

The design of an efficient sampling rate conversion system is a major problem for the multirate digital signal processing. The filtering processing takes the most of computational time in sampling rate conversions. The computational efficiency is increased by such technologies as the polyphase filtering and the fast Fourier transform (FFT) algorithm [1]. The basic idea of the polyphase implementation is to commute the filtering operation to a low sampling rate whereby computational efficiency is achieved. Since signals are real in most of the practical applications, it is important to search for efficient implementation of sampling rate conversion systems for real sequences. Real-valued fast algorithms are investigated in the literature [2]. Let $x(n)$ be a real sequence of length N, and the sequence is represented

Manuscript received February 24, 1995; revised September 16, 1996. The associate editor coordinating the review of this paper and approving it for publication was Prof. Tyseer Aboulnasr.

The author is with the Kanazawa Institute of Technology, Ishikawa, Japan.

Publisher Item Identifier S 1053-587X(97)02571-3.