Optimal signaling for detection in doubly dispersive multipath

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***Abstract*—In this paper, we revisit the problem of signal detection in multipath scattering. Specifically, we explore the effect of bandwidth and signaling duration on non-coherent signal detection in doubly dispersive non-uniform multipath scattering environments. In such environments, we conjecture that detection performance is optimized if two conditions are met: *C1*) the number of independent degrees of freedom induced by the transmit signal on the channel is approximately $\mathrm{SNR}/2$, and *C2*) the variance of the channel coefficients (each corresponding to a degree of freedom) is equal. In uniform multipath, the second condition is inherent. In non-uniform scattering, the variance of the channel coefficients depends on the transmit signal which critically impacts detection performance. As such, non-uniform doubly selective channels afford us an additional design parameter; we can design packets for detection of a specific *aspect ratio*, the ratio of bandwidth to signaling duration. If our packets meet the two criteria above, numerical and analytical results indicate detection performance is maximized. When the above conditions cannot be met, we present an algorithm for finding the signaling duration and bandwidth that maximize detection performance.**

I. Introduction

Signal detection in multipath is not a new problem; early work [1] as well as recent work [2], has considered detection performance in frequency selective, uniform scattering multipath environments. In this work we consider detection performance in doubly selective, uniform and non-uniform scattering. Our focus is on maximizing the probability of signal detection, viewed as an optimization over all values of signaling duration, T, and bandwidth, W, within design constraints. The main contribution of this work is to *i)* present conditions under which detection probability is maximized, *ii)* present an algorithm for achieving these conditions, and *iii)* present supporting analytical and numerical results.

Figure 1 illustrates three packets used for detection. Each packet has a different signaling duration, T, and bandwidth, W. The area of each packet represents total resources used by the packet - the time bandwidth product, or signal space dimension. In a given multipath environment, what signaling duration and bandwidth maximize detection performance? At a high level, we pose this question as an optimization problem:

$$\begin{aligned} &\max_{T,W} \quad P_D && (1)\\ &\text{subject to} \quad P_{FA} = \text{constant} \\ &\qquad\qquad\quad \mathrm{SNR} = \text{constant} \\ &(optional) \qquad N = \text{constant} \end{aligned}$$

where P_D is the probability of detection, P_{FA} the probability of false alarm, SNR the received signal to noise ratio, and $N = TW$ the time bandwidth product.

TABLE I
Optimal Packet Configurations

	Delay	Doppler	Optimal Packet Configuration
I	Uniform	Uniform	c_1, c_2, or c_3
II	Non-uniform	Uniform	c_3
III	Uniform	Non-uniform	c_1
IV	Non-uniform	Non-uniform	*Use algorithm* 1

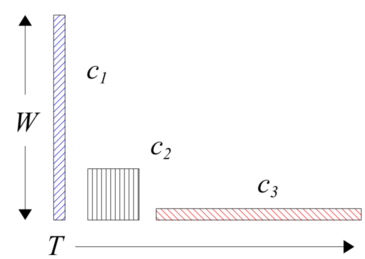


Fig. 1. Three packets used for detection. While the time-bandwidth product (area) of each packet is the same, the *aspect ratio* is different, critically impacting detection performance. The colors of the packets correspond to the color of the plots in Figures 5 - 9.

As we elaborate in Section II, T and W control both the number and variance of the statistically independent degrees of freedom (DoF) induced by the channel on the received signal. As we increase the DoF, we have two competing effects: *i)* resistance to multipath fading increases (the probability that all channels fade simultaneously diminishes) and *ii)* signal to noise ratio *per* DoF decreases. These competing effects result in optimal probability of detection for some (not necessarily unique) T and W.

Figure (2) illustrates the interaction between channel parameters, design parameters, and detection performance. For a given multipath environment, changing the design parameters, primarily T and W, critically impacts detection performance.

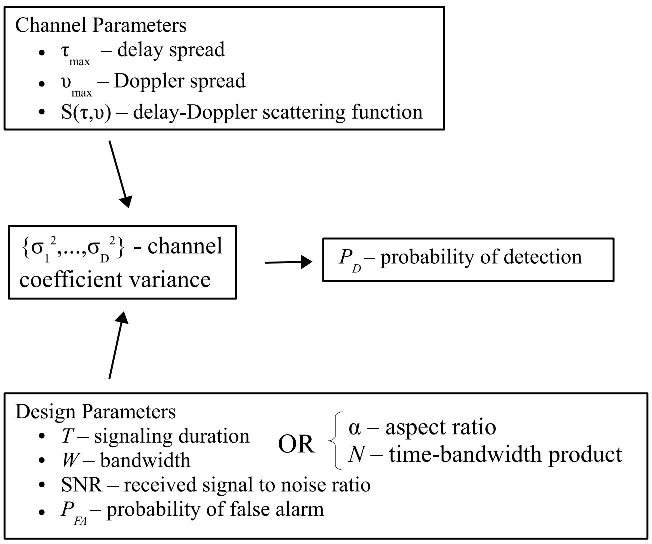


Fig. 2. Diagram showing the interaction between the channel parameters and the system design parameters. The channel parameters as well as the design parameters define the power of the channel coefficients – which in turn, dictate detection performance.

As we conclude in this paper, equation (1) is optimized (P_D is maximized) if two conditions are met:

1) the number of statistically independent degrees of freedom induced by the transmit signal on the channel are approximately equal to $\frac{SNR}{2}$, and
2) the variance of the channel coefficients is equal.

Table I and Figure 1 show packet configurations that can potentially achieve these criteria in different multipath scattering environments. If the scattering environment is non-uniform both delay and Doppler (case IV) it may not be possible satisfy the second requirement. Under these conditions, we present an efficient algorithm to search for the optimal packet configuration.

Signal detection in multipath environments is an important problem in a number of applications: non-coherent communications, detection problems in sensor networks, global positioning systems, and radar. As carrier frequencies increase, coherent communication often becomes impractical – non-coherent detection becomes increasingly important. Signal detection in multipath arises in the so called *interweave* paradigm in cognitive radio. In the *interweave* paradigm, unlicensed spectrum users opportunistically use licensed spectrum when primary transmitters are not active [3]. In order to facilitate detection, licensed users transmit a 'beacon' signal to warn that the spectrum is occupied. What type of transmit beacon maximizes the probability of detection at a receiver? This depends on the multipath channel - if we know the statistics of the channel, we can design a detection packet to meet a specification with minimal resources – or conversely, given a resource constraint, we can design a packet to maximize chances of detection.

The remainder of this paper is organized into 5 sections. In Section II, we define the model for multipath channel, the received signal, the detector, and discuss statistics that dictate detection performance. Most importantly, we highlight the interaction between the transmit signal and the multipath environment. In Section III we begin with a general overview of problem and conclusions from the work. In Section IV and V, we explore analytical and numerical results – and how they support the conjectures. Section VI concludes by reviewing the most important aspects of the work.

II. System Model

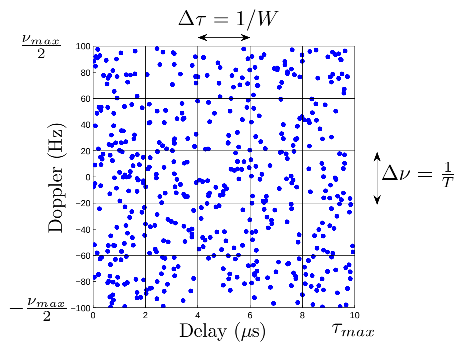


Fig. 3. Illustration of the virtual channel representation. Each dot represents a multipath component. Each square represents a delay-Doppler resolution bin of size $\Delta\tau \times \Delta\nu$, corresponding to a channel coefficient, h_d. L is the number of bins in delay, while M is the number of bins in Doppler. The channel coefficient is approximately equal to the sum of the paths in the corresponding bin [4].

A. Virtual Channel Representation

We begin with a model for the received signal in doubly selective multipath presented in [5] [6] [7] [2]. The model assumes a RAKE type receiver. The D dimensional received signal can be written as

$$\mathbf{r} = \sqrt{\mathcal{E}}\mathbf{h} + \mathbf{w} \quad (2)$$

where $\mathcal{E}$ denotes the transmit signal energy, $\mathbf{w} \in \mathbb{C}^D$ additive gaussian noise, and $\mathbf{h} \in \mathbb{C}^D$ the D dimensional vector of channel coefficients. The channel coefficients, $\{h_d\}$, represent the Fourier series expansion of the time varying frequency response of the channel, $H(t, f)$, restricted to $(t, f) \in [0, T] \times [-W/2, W/2]$:

$$H(t, f) = \sum_{\ell=1}^{L} \sum_{m=-(M-1)}^{M-1} h_{\{\ell,m\}} e^{j2\pi \frac{m}{T} t} e^{-j2\pi \frac{\ell}{W} f}. \quad (3)$$

The number of coefficients required to accurately represent channel with a Fourier series expansion depends on the signaling duration, T, bandwidth, W, as well as the delay spread, τ_{max}, and Doppler spread, ν_{max}, of the channel as:

$$D = L(2M - 1) \quad L = \lceil W\tau_{max} \rceil \quad M = \lceil T\nu_{max}/2 \rceil . \quad (4)$$

The number of channel coefficients, D, also represents the statistically independent degrees of freedom (DoF) induced by the transmit signal on the multipath channel. L represents the number of bins, or channel partitions, in delay, while $2M-1$ is the number of resolutions bins in Doppler. This can be visualized by referencing Figure 3; as bandwidth and signaling duration are increased, the number of bins, D, increases. A detailed proof is given in [4], chapter 5.

We define the normalized packet *aspect ratio*, α, as the ratio of bandwidth to signaling duration scaled by the channel parameters:

$$\alpha = \frac{W\tau_{max}}{T\nu_{max}}. \tag{5}$$

When $\alpha = 1$, the channel has an equal number of partitions in delay and Doppler, thus $L \approx 2M-1$. When α is small, the packet is long in duration and narrow in bandwidth; conversely, if α is large, the packet is wide in bandwidth and short in duration.

B. Statistics

As discussed in [4], by virtue of path partitioning, the channel coefficients are distributed as independent complex gaussian random variables:

$$\begin{aligned} \mathbf{h} &\sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma_h}) \\ \mathbf{\Sigma_h} &= E[\mathbf{h}\mathbf{h}^H] = \mathrm{diag}(\sigma_1^2, ..., \sigma_{\mathrm{D}}^2) = \mathrm{diag}(\boldsymbol{\sigma}^2). \end{aligned} \tag{6}$$

The variance of each channel coefficient, $\{\sigma_d\}$, depends on the scattering environment and L and M. In general, the delay-Doppler scattering function (DDSF), $S_c(\tau, \nu)$, describes the variance of the time-varying channel impulse response as a function of delay and Doppler [8]; the approximate variance of the channel coefficient is found by evaluating the scattering function at the center of each delay-Doppler resolution bin.

We focus on detection performance in four illustrative, separable multipath environments: uniform in delay and Doppler, exponential in delay and uniform in Doppler, uniform in delay and 'bathtub' [9] in Doppler, and exponential in delay and 'bathtub' in Doppler. Table II defines the variance of the D channel coefficients, where $\bar{\mu}$ is a constant equal to the average delay spread, $\ell \in \{1, 2, ..., L\}$ is the index of the channel partition in delay, and $m \in \{1, 2, ..., 2M-1\}$ is the index of the channel partition in Doppler.

TABLE II
MULTIPATH ENVIRONMENTS

Uniform Delay - Uniform Doppler
$\sigma_d^2(m,\ell) = \frac{1}{L(2M-1)}$
Exponential Delay - Uniform Doppler
$\sigma_d^2(m,\ell) = \frac{1}{\bar{\mu}} e^{-\tau_{max}\frac{(\ell-1/2)}{L\bar{\mu}}}$
Uniform Delay - Bathtub Doppler
$\sigma_d^2(m,\ell) = \frac{2}{\pi\nu_{max}} \frac{1}{\sqrt{1-(\frac{2m-1}{M}-1)^2}}$
Exponential Delay - Bathtub Doppler
$\sigma_d^2(m,\ell) = \frac{2}{\pi\bar{\mu}\nu_{max}} \frac{e^{-\tau_{max}\frac{(\ell-1/2)}{L\bar{\mu}}}}{\sqrt{1-(\frac{2m-1}{M}-1)^2}}$

We cast our signal detection problem as a binary hypothesis test. H_0 will be defined as the noise only hypothesis; H_1 as the signal plus noise hypothesis. The received signal can be written as

$$\begin{aligned} H_0 &: \quad \mathbf{r} = \mathbf{w} & \sim \mathcal{CN}(\mathbf{0}, N_0\mathbf{I}) \\ H_1 &: \quad \mathbf{r} = \sqrt{\mathcal{E}}\mathbf{h} + \mathbf{w} & \sim \mathcal{CN}(\mathbf{0}, \mathcal{E}\mathbf{\Sigma_h} + N_0\mathbf{I}). \end{aligned} \tag{7}$$

where $\mathcal{E}$ is the transmit signal energy and N_0 the noise power. We assume the variance of the channel coefficients is known or estimated. The optimal detector is a threshold test with test statistic Z of quadratic form [2]:

$$Z = \sum_{d=1}^{D} |r_d|^2 \left(\frac{1}{N_0} - \frac{1}{\mathcal{E}\sigma_d^2 + N_0} \right). \tag{8}$$

The decision is based on a threshold γ: if $Z > \gamma$ the detector decides H_1, otherwise, the detector decides H_0. The test statistic under either hypothesis is a sum of exponential random variables (equivalently, a sum of chi-squared random variables each with 2 degrees of freedom). Z has a *hypo-exponential* distribution [2] [10]:

$$\begin{aligned} H_0 &: \quad Z \sim \mathrm{Hypo}\left(1 + \frac{1}{\mathrm{SNR}_1}, ..., 1 + \frac{1}{\mathrm{SNR}_D}\right) \\ H_1 &: \quad Z \sim \mathrm{Hypo}\left(\frac{1}{\mathrm{SNR}_1}, ..., \frac{1}{\mathrm{SNR}_D}\right) \end{aligned} \tag{9}$$

where the *per channel* signal to noise ratio (SNR_d) is

$$\mathrm{SNR}_d = \frac{\mathcal{E}\sigma_d^2}{N_0} \qquad \sum_{d=1}^{D} \mathrm{SNR}_d = \mathrm{SNR}. \tag{10}$$

The *hypo-exponential distribution* has a cumulative density function which can be expressed in closed form as [10]

$$P(Z < z) = F(z) = 1 - \mathbf{e_1}^T e^{z\mathbf{G}} \mathbf{1} \quad , \quad z \geq 0 \tag{11}$$

where

$$\mathbf{G} = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & -\lambda_{D-1} & \lambda_{D-1} \\ 0 & \cdots & 0 & 0 & -\lambda_D \end{bmatrix}_{[D \times D]} \tag{12}$$

$$\mathbf{1} = [1 \;\; 1 \;\; ... \;\; 1]^T_{[D \times 1]} \qquad \mathbf{e_1} = [1 \;\; 0 \;\; ... \;\; 0]^T_{[D \times 1]}, \tag{13}$$

$\{\lambda_d\}$ are the parameters of the distribution (from equation (9), $\lambda_d = \frac{1}{\mathrm{SNR}_d}$ under H_1), and e is the matrix exponential operator. We assess the performance of the detectors based on the error probabilities: the probability of false alarm, P_{FA}, and the probability of missed detection, P_{MD}:

$$P_{FA} = P(Z > \gamma | H_0) \qquad P_{MD} = P(Z < \gamma | H_1). \tag{14}$$

III. Overview of Results

Equation (1) is an optimization over two variables; using the framework of the previous section, an exhaustive search over all $\{T,W\} \in \left[\frac{1}{W}, T_{max}\right] \times \left[\frac{1}{T}, W_{max}\right]$ could be employed. Note T_{max} and W_{max} are the maximum values allowable values of T and W dictated by design constraints. The lower bound is imposed by time-bandwidth duality – T cannot be less than $1/W$, and W less than $1/T$.

It is insightful to transform the optimization over T and W in (1) to a larger optimization over the variance of the channel coefficients (or equivalently, the signal to noise ratios of the channel coefficients). If we can solve the larger optimization problem, and show there exist a T and W that induce the same optimal, we claim to have solved the optimization.

Using the closed form expression of the hypo-exponential distribution (11), we can re-write (1) as an optimization over the signal to noise ratios of the individual channel coefficients. We seek to maximize the probability of signal detection ($1 - P_{FA}$):

$$\max_{\{\mathrm{SNR}_d\} \in \mathbb{R}^n} \quad \mathbf{e_1}^T e^{\gamma \mathbf{G_1}} \mathbf{1} \tag{15}$$
$$\text{subject to}$$
$$\gamma : 1 - \mathbf{e_1}^T e^{\gamma \mathbf{G_0}} \mathbf{1} = P_{FA}$$
$$\sum_{d=1}^{n} \mathrm{SNR_d} = \mathrm{SNR}$$

where $\mathbf{e_1}_{[n \times 1]}$ and $\mathbf{1}_{[n \times 1]}$ are defined in equation (12), $\mathbf{G}_0$ and $\mathbf{G}_1$ defined by equation (12) under H_0 and H_1 respectively, SNR and P_{FA} are constants, and n a positive integer.

We conjecture, based on analytical results and insight gained from numerical results, that the above problem is optimized (the probability of detection is maximized) when:

$$\mathrm{SNR}_d \approx \begin{cases} \frac{1}{c} & d = 1, ..., \lceil c \cdot \mathrm{SNR} \rceil \\ 0 & d = (\lceil c \cdot \mathrm{SNR} \rceil + 1), ..., n \end{cases} \tag{16}$$

or equivalently,

- *condition C1)* the number of independent degrees of freedom[1], D, induced by the transmit signal on the channel is approximately $D_{opt} = c \cdot \mathrm{SNR}$, where $c \approx 1/2$ with a small dependance on P_{FA}, and
- *condition C2)* the variance of the channel coefficients (each corresponding to a degree of freedom) is approximately equal.

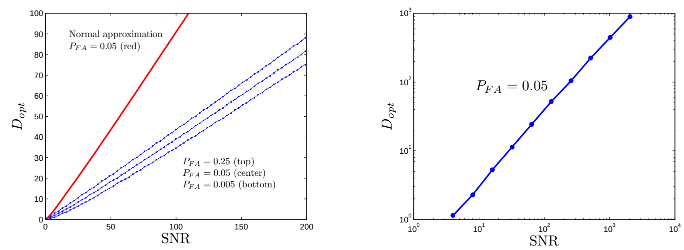
If there exists a T and W such that $\{\mathrm{SNR}_d\}$ satisfy the above conditions (or equivalently equation 16), we conjecture this T and W maximize detection probability.

Table I lists configurations that potentially achieve these conjectures – if the channel is uniform in either delay or Doppler, we spread our transmit energy along that domain (see equation (4)) to satisfy condition *C2)*. As we explore in Section V-D, in channels that exhibit non-uniform scattering in both delay and Doppler, there may not exist a T and W that satisfy the above conditions. In this case, we search for a T and W that come close to satisfying the two optimality conditions.

[1]Note that the number of independent degrees of freedom, D, is equivalent to the number of *non-zero* channel coefficients.

Summary of support for condition C1): Figure 4(a) shows optimal degrees of freedom as a function of SNR for a uniform multipath environment for three values of P_{FA}. While there is a small dependance on P_{FA}, the optimal number of degrees of freedom, D_{opt} is approximately $\frac{\mathrm{SNR}}{2}$. Numerical results throughout the following section further support this conjecture. Additionally, a normal approximation for large D indicates D_{opt} has a near linear dependance on the SNR (see Section IV).

Summary of support for condition C2): Figures 6(d) - (f) and Figures 9(d) - (f) show bar graphs of the channel coefficient variance, and the corresponding detection probabilities. The probability of missed detection is minimized when the channel coefficients have equal power. When the channel coefficients are perturbed by a small amount, numerical results indicate detection performance decreases (see Figures 5(b) - 9(b)). Additionally, equation (15) is a *symmetric* function of $\{\mathrm{SNR}_d\} \in \mathbb{R}^n$, which implies existence of a stationary point at $\mathrm{SNR}_1 = \mathrm{SNR}_2 = ... = \mathrm{SNR}_D$ as defined in equation (16) (see proof in Section IV).



(a) (b)

Fig. 4. (a) Signal to noise ratio plotted against optimal degrees of freedom for three values of P_{FA} in uniform multipath generated using the closed form of the *hypo-exponential* distribution. While the figure shows a small dependance on the probability of false alarm, we approximate D_{opt} as $\mathrm{SNR}/2$. (b) Signal to noise ratio plotted against optimal degrees of freedom on a log-log scale.

IV. Analytical Support

In this section, we present analytical results to support the conjecture that detection performance is optimized when conditions *C1* and *C2* are met.

A. Uniform Multipath - Condition C1

We begin with support for condition *C1*. We assume that condition *C2* is met (in uniform multipath, condition *C2* is inherent). A similar problem has been studied in [1], chapter 14, in the context of minimum probability of error over a communications channel. The authors of [1] arrive at a simple rule of thumb for optimal degrees of freedom - $D_{opt} \approx \frac{\mathrm{SNR}}{3}$.

If the signal to noise ratio of the individual channel coefficients is equal, the expression in equation (15) can be written in a simpler form:

$$\min_D P_{MD} = \min_D F_{2D}\left(F_{2D}^{-1}(1-P_{FA})\frac{D}{\mathrm{SNR}+D}\right) \quad (17)$$

where D is the number of non-zero channel coefficients, F_{2D} is the chi-squared cumulative density function with $2D$ degrees of freedom, and F_{2D}^{-1} is the inverse chi-squared CDF. In order to simplify the expression further, we look at the specific case where $P_{FA} = 0.5$. In this case, we have that $F_{2D}^{-1}(1-P_{FA}) \approx 2D$ (the median rapidly approaches the mean). We have

$$\begin{aligned} P_{MD} &\approx F_{2D}\left(\frac{2D^2}{\mathrm{SNR}+D}\right) \quad (18)\\ &= \frac{\int_0^{\frac{D^2}{\mathrm{SNR}+D}} t^{D-1}e^{-t}dt}{\int_0^{\infty} t^{D-1}e^{-t}dt} \end{aligned}$$

where the second equality follows from the definition of the chi-squared distribution. Equation (18) is known as the *incomplete gamma function* [11] and can be numerically evaluated using standard mathematical software.

We can approximate equation (17) using a normal approximation: $F_{2D}(x) \to 1 - Q\left(\frac{x-2D}{2D^{1/2}}\right)$. Re-writing equation (17), we have

$$\max_D Q\left(\frac{Q^{-1}(P_{FA})D + \mathrm{SNR}\sqrt{D}}{\mathrm{SNR}+D}\right) \quad (19)$$

where $Q(x) = \int_x^{\infty} \frac{1}{2\pi}e^{-t^2}dt$. Equation (19) can be maximized by differentiating the argument of the Q function:

$$\begin{aligned} D_{opt} &= \mathrm{SNR} + 2Q^{-1}(P_{FA}) \quad (20)\\ &\quad -2Q^{-1}(P_{FA})\sqrt{Q^{-1}(P_{FA})^2 + \mathrm{SNR}}\\ D_{opt} &\approx \mathrm{SNR}. \end{aligned}$$

Figure 4 show the normal approximation plotted with numerical minimization of equation (17). While the normal approximation does capture the linear trend, it does not well estimate the true optimal degrees of freedom.

B. Non-uniform Multipath - Condition C2

In non-uniform multipath, the signal to noise ratios of the individual channel coefficients can take on distinct values, and we return to the optimization as stated in equation (15). Equation (15) is not a convex function of $\{\mathrm{SNR}_1 ... \mathrm{SNR}_n\}$. Numerical evaluation of the case where $n = 2$ and $\mathrm{SNR}_1 + \mathrm{SNR}_2 = 1/2$ shows the function, in general, is not convex.

The function in equation (15) is a symmetric function[2]. This implies that the solution $\{\mathrm{SNR}_1 = \mathrm{SNR}_2 = ... = \mathrm{SNR}_D\}$ is a stationary point.

Sketch of Proof: Consider a symmetric function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, with constraint $\mathbf{x}^T\mathbf{1} = 1$, as we have in equation (15). *Lemma: The hyperplane defined by the constraint* $\mathbf{x}^T\mathbf{1} = 1$ *is perpendicular to* $\mathbf{x} = a\mathbf{1}$. Let $U_{[n\times n]}$ be a permutation matrix with exactly one 1 in each row and each column. By definition of a symmetric function, $f(\mathbf{x}) = f(U\mathbf{x})$. Using the chain rule (where $\nabla f(\mathbf{x})$ is the gradient, or vector of partial derivatives), we have:

$$\nabla f(U\mathbf{x}) = U\nabla f(\mathbf{y})|_{\mathbf{y}=U\mathbf{x}} \quad (21)$$

Restricting $\mathbf{x} = a\mathbf{1}$ (i.e., $x_1 = x_2 = ... = x_n$), we have $U\mathbf{x} = \mathbf{x}$ and equation (21) is then

$$\nabla f(\mathbf{x}) = U\nabla f(\mathbf{x}) \quad (22)$$

which is true for all permutation matrices, U, if and only if

$$\nabla f(\mathbf{x})|_{\mathbf{x}=a\mathbf{1}} = b\mathbf{1}. \quad (23)$$

The gradient of the function $f(\mathbf{x})$ along the vector $\mathbf{x} = a\mathbf{1}$ is perpendicular to hyperplane defined by the constraint, and $\mathbf{x} = a\mathbf{1}$ is a stationary point [12].

Note that fixing any $\mathrm{SNR}_d = 0$ is equivalent to reducing the dimensionality of the optimization in equation (15). When the dimensionality is reduced so that $n = D_{opt}$, the conditions in equation (16) indicate the gradient of the function is perpendicular to the constraint.

V. Numerical Support

In this section we compare detection performance as a function of signaling duration T, and signal bandwidth W. The channels are assumed to be rich in multipath (i.e, each delay-Doppler bin contains many discrete paths, for all signaling durations and bandwidths); moreover, $\tau_{max} = 10\mu$s, $\nu_{max} \approx 200$Hz, $P_{FA} = 0.05$, and $\mathrm{SNR} = 100$.

For each scattering environment, the variance of the D channel coefficients, $\{\sigma_d^2\}$, are calculated using the expressions found in Table II. They are normalized as

$$\sum_{d=1}^{D}\sigma_d^2 = 1 \qquad \mathrm{SNR} = \frac{\mathcal{E}}{N_0} = \sum_{d=1}^{D}\frac{\mathcal{E}\sigma_d^2}{N_0} = \sum_{d=1}^{D}\mathrm{SNR_d}. \quad (24)$$

For a given P_{FA} the cumulative density function of the test statistic under H_0 can be inverted numerically to find the corresponding threshold γ. The detector performance is assessed via P_{MD} for the corresponding threshold. P_{MD} is calculated using the closed-form expression for the *hypo-exponential* distribution in (11).

Figures 5(a), 6(a),7(a), and 9(a) show detection performance as a function of D, the degrees of freedom of the channel, under the four different DDSFs. Each graph contains three plots of P_{MD} as defined in Table III. The three plots represent different ways to increase D: (i) by increasing W (partitioning in delay), (ii) increasing T (partitioning in Doppler), and (iii) by increasing T and W proportionally (partitioning in both delay and Doppler). Figure (1) illustrates these three approaches - packet c_1 is wide in bandwidth, c_2 is proportional in both signaling duration and bandwidth, and c_3 is long in signaling duration. The total area (time-bandwidth product) of each packet is the same, yet the *aspect ratio* of the packets are different.

Figures 5(b), 6(b), 7(b), and 9(b) show detection performance as a function of the packet *aspect ratio*, α. Each figure contains three plots of P_{MD}, corresponding to different

[2] A symmetric function is a function that is invariant to permutations of the input, i.e. for $x, y \in \mathbb{R}$, $f(x, y) = f(y, x)$

TABLE III
CHANNEL PARTITIONING IN FIGURES 5 - 9(A)

Increasing W - blue (dashed)		
$W \propto D$	$D = L = \lceil W\tau_{max} \rceil$	
Increasing T - red (solid)		
$T \propto D$	$D = (M) = \lceil T\nu_{max}/2 \rceil$	
Increasing W and T - black (dotted)		
$W, T \propto \sqrt{D}$	$L = (M)$	$\frac{T}{W} \approx \frac{\tau_{max}}{\nu_{max}}$

fixed time-bandwidth products: *i)* solid - $TW = 5 \times 10^3$, *ii)* dashed - $TW = 2 \times 10^4$, *iii)* dotted - $TW = 5 \times 10^4$. The horizontal plots labeled $P_{MD_{min}}$ show the minimum attainable probability of missed detection for all values of D, α and $\{\sigma_d^2\}$. In all cases $P_{MD_{min}} = 2.982 \times 10^{-8}$. This is considered our global minimum P_{MD} for all channels with $\text{SNR} = 100$ and $P_{FA} = 0.05$.

Figures 5(c), 6(c), 7(c), and 9(c) illustrate representative multipath environments corresponding to the figures directly above. Each dot represents a discrete multipath component.[3] The power of the multipath component is represented by the area of the dot. The solid lines in each figure show different ways in which the channel can be partitioned.

A. Uniform Multipath

Figure 5 shows detection performance under uniform multipath. As described above, Figure 5(a) plots detection performance as a function of D for packets of three different *aspect ratios*. For a fixed D, P_{MD} does not depend on the *aspect ratio* of the packet - all three plots in fig. 5(a) are identical. Regardless of how the multipath environment is partitioned, the variance of the D channel coefficients are identical. Performance is constant for fixed D.

For all three plots in 5(a), the optimal performance occurs at approximately $D = 40$. As noted in [2], this optimal occurs at $D = D_{opt} \approx \frac{SNR}{2}$ - optimal detection performance occurs when the signal to noise ratio per degree of freedom is approximately 2 – see Figure 4(a).

Figure 5(b) shows detection performance as a function of packet *aspect ratio*. The left of the graph corresponds to packets that are small in bandwidth and long in duration. The right corresponds to packets that are wide in bandwidth and short in duration. Detection performance deteriorates rapidly when the aspect ratio becomes large or small. As this happens, D becomes large (even though the time bandwidth product is constant). D exceeds $\frac{SNR}{2}$ and detection performance deteriorates. For extreme values of α, D exceeds D_{opt} at a linear rate.

As an example, consider the dashed line in Figure 5(b). What occurs when the packet aspect ratio exceeds 1×10^2? As we continue to increase the aspect ratio, we increase W - thus L increases. Signaling duration decreases but M, bounded below by 1, does not decrease. This causes D to increase linearly with W; we exceed optimal D and detection performance deteriorates.

[3] Even though there are a finite number of multipath components in each figure, we assume that for all signaling dimensions evaluated, there are sufficiently many paths in each delay-Doppler bin so that the central limit theorem can be invoked - this is known as *rich* multipath.

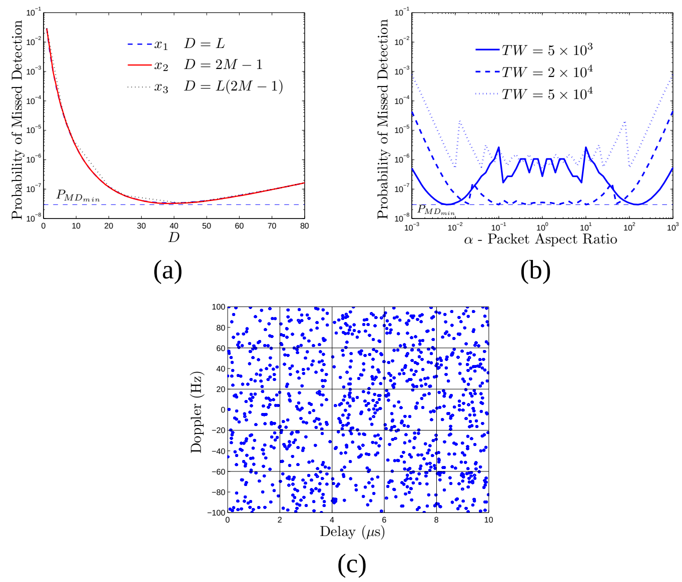


Fig. 5. Uniform delay-Doppler scattering environment - (a) P_{MD} as function of D for three *aspect ratios* as described by Table III. (b) P_{MD} as a function of packet *aspect ratio* for three values of TW. The horizontal plot shows the global minimum probability of missed detection, $P_{MD} = 2.982 \times 10^{-8}$ (c) Illustration of a representative physical multipath environment. $P_{FA} = .05$, $\text{SNR} = 100$.

B. Exponential Delay

Unlike uniform DDSFs (for D fixed), in an exponential delay, uniform Doppler environment, detection performance depends on L and M independently, instead of their product. Fig. 6(a) shows detection performance as a function of D of an exponential delay, uniform Doppler multipath channel, such as that illustrated in fig. 6(c). The red (solid) plot in the figure represents performance when D is increased by increasing the signaling duration of the transmit signal and keeping bandwidth fixed. This partitions the multipath environment in Doppler (indicated by the solid lines in Figure 6(c)). Since the multipath environment is uniform in Doppler, partitioning in Doppler only scales the variance of the channel coefficients (see Table II); all channel coefficients will have equal variance (see Figure 6(d)). Empirically, optimal performance is reached when each channel coefficient has equal variance approximately equal to two - achieved by increasing T.

The dashed (blue) plot in fig. 6(a) shows performance when D is increased by increasing bandwidth and holding signaling duration constant. This partitions the channel in delay. Some of the channel coefficients (corresponding to longer delays) will contain little power as D is increased (see Figure 6(f)) – optimal performance will not be reached since the variance of each channel coefficients is not equal.

Figure 6(b) shows detection performance as a function of α for three time-bandwidth products. For extreme aspect ratios, detection performance decreases for the same reasons discussed in the uniform case. For aspect ratios between approximately 10^{-2} and 10^{2}, there is a tilt to the plot – detection performance is better for packets that are longer in duration and short in bandwidth. As discussed above, this induces a uniform power distribution on the channel coefficients.

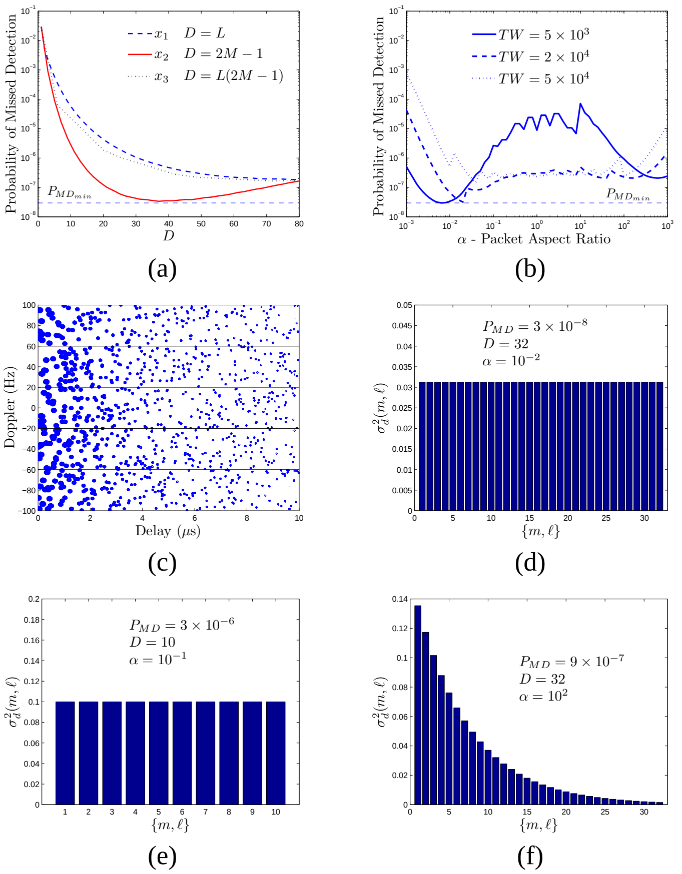


Fig. 6. Exponential delay - uniform Doppler environment - (a) P_{MD} as function of D for three *aspect ratios* as described by Table III. (b) P_{MD} as a function of packet α for three values of TW. (c) Illustration of a representative physical multipath environment. $P_{FA}=.05$, $\text{SNR}=100$. (d) - (f) Variance of the channel coefficients when probed with by three signals of different α. (d) Probed by a near optimal signal ($TW=5\times10^{3}$ and $\alpha=10^{-2}$). (e) Sub-optimal approach with $TW=5\times10^{3}$, but $\alpha=10^{-1}$ resulting in $D\neq\frac{SNR}{2}$. (f) Sub-optimal approach where $TW=5\times10^{3}$ and $\alpha=10^{2}$.

C. Bathtub Doppler

Figure 7(a) and 7(b) plot detection performance for the 'bathtub' Doppler - uniform delay profile illustrated in Figure 7(c). Increasing bandwidth most effectively partitions the channel - for the same reasons as discussed in exponential DDSF case but in Doppler instead of delay. Since the channel is uniform in delay, partitioning in delay results in channel coefficients with equal variance; optimal performance is reached when $\text{SNR}/D\approx2$. For aspect ratios between approximately 10^{-2} and 10^{2}, there is a small tilt to the plot - opposite the tilt in Figure 6(b). Detection performance is better for packets that are wide in bandwidth.

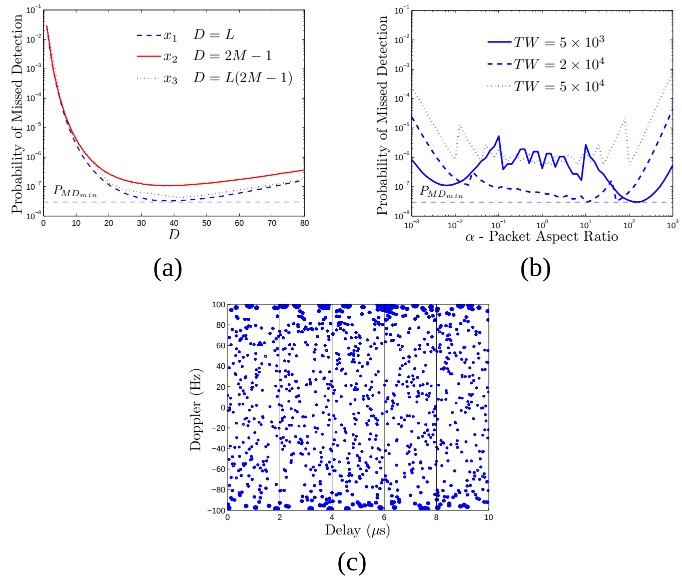


Fig. 7. Uniform delay - 'bathtub' Doppler - (a) P_{MD} as function of D for three *aspect ratios* as described by Table (III). (b) P_{MD} as a function of packet *aspect ratio* for three values of TW. (c) Illustration of a representative physical multipath environment. $P_{FA}=.05$, $\text{SNR}=100$.

D. Exponential Delay, Bathtub Doppler

Figure 9(a) and 9(b) show performance of a channel with exponentially distributed delay and 'bathtub' Doppler. $P_{MD_{min}}=2.982\times10^{-8}$ (plotted horizontally) is not achieved. Regardless of how we partition the channel, we do not induce uniform power in the channel coefficients (see Figure 9(d)-(f)) – a requirement to achieve optimal performance.

This specific case highlights a general class of multipath environments in which diversity is non-uniform in both delay and Doppler. Under the constraints of our model, the channel cannot be partitioned so that variance of the channel coefficients are equal.

In order to find an optimal signaling duration and bandwidth, we present an algorithm to minimize P_{MD}. While equation (1) suggests a search over all $\{T,W\}$, it is advantageous to convert the optimization to a search over $\{\alpha,TW\}$, an equivalent problem. The algorithm converts the two dimensional optimization into a two step, one variable, recursive optimization.

Algorithm 1 P_{MD} Optimization

$TW_{opt}=$ start value
while $|P_{MD}^{k}-P_{MD}^{k-1}|>$tolerance **do**
  calculate $\alpha_{opt}=\underset{\alpha}{\text{argmin}}\quad P_{MD}(\alpha,TW_{opt})$
  calculate $TW_{opt}=\underset{TW}{\text{argmin}}\quad P_{MD}(\alpha_{\text{opt}},TW)$
  $P_{MD}^{k-1}=P_{MD}^{k}$
  $P_{MD}^{k}=P_{MD}(\alpha_{\text{opt}},TW_{opt})$
end while
return TW_{opt}, α_{opt}, P_{MD}^{k}

Algorithm 1 first fixes a TW and then searches over all

α. When an optimal α is found, the algorithm then searches over all TW. This processes is repeated until the algorithm converges on an optimal P_{MD}. This approach offers many advantages to a search over $\{T, W\}$. Primarily, $\{\alpha, TW\}$ provide a natural basis to achieve the conjectured optimality conditions in Section I; the algorithm often converges in as few as two steps. Additionally, for a fixed α, P_{MD} appears to be a convex function of TW.

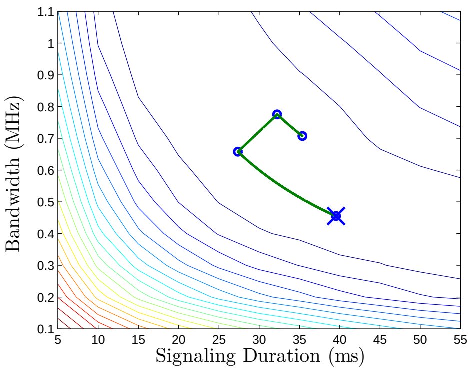


Fig. 8. Contour plot of P_{MD} with a visualization of algorithm 1. The algorithm starts by searching along a line of constant TW, then along a constant α, until converging on an optimal T and W.

VI. Conclusion

In this paper, we explored the effect of signaling duration and bandwidth on non-coherent detection performance in doubly selective multipath. Specifically, we attempted to maximize the probability of signal detection, as a function of signaling duration and bandwidth, for a given multipath environment.

Numerical and analytical results suggest that optimal performance is reached when the following two conditions are met:

1) the channel is partitioned so the channel coefficients have equal variance, and
2) the number of independent degrees of freedom, D is approximately $\frac{SNR}{2}$.

In exponential delay, uniform Doppler environments, this suggests we send packets that are long in duration and narrow in bandwidth. In uniform delay and 'bathtub' Doppler, this suggests we use detection packets that are wide in bandwidth, and short in duration. If the channel is non-uniform in both delay and Doppler, the developments of this paper can be used to find the signaling duration and bandwidth that result in best performance.

Acknowledgements

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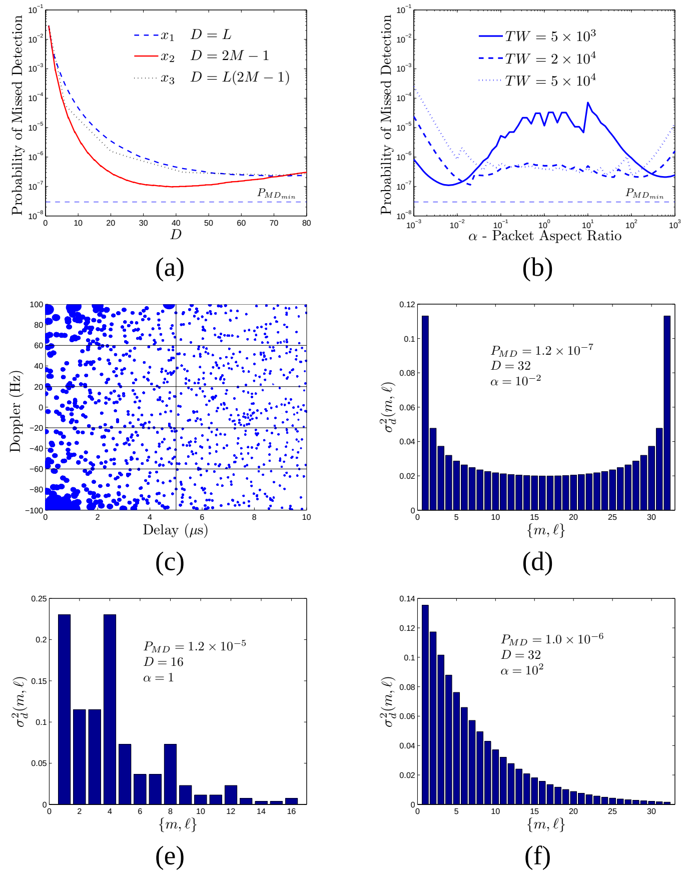

Fig. 9. Exponential delay - 'bathtub' Doppler environment - (a) P_{MD} as function of D for three *aspect ratios* as described by Table III. (b) P_{MD} as a function of packet *aspect ratio* for three values of TW. (c) Illustration of a representative physical multipath environment. $P_{FA} = .05$, SNR $= 100$. (d) - (f) Variance of the channel coefficients when probed with packets of different *aspect ratio*. $TW = 5 \times 10^{-3}$. (d) corresponds to partitioning in Doppler, (e) in both delay and Doppler, and (f) in delay.

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