Exploiting Time-Frequency Coherence to Achieve Coherent Capacity in Wideband Wireless Channels

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Abstract

It was widely believed that knowledge of channel state information (CSI) at the receiver imposes a sharp cut-off on the achievability of coherent capacity at large bandwidths (or low SNRs). Recent works have shown that by either employing an explicit training-based scheme or an implicit channel-learning and communication scheme, rates intermediate between the coherent and the non-coherent extremes can be achieved. However, to bridge the gap between these two extremes, these works assume that the coherence time of the channel increases as the signaling bandwidth increases, without providing any physical basis that could lead to such a scaling relationship. In this paper, we study the wideband capacity of doubly dispersive underspread wireless channels employing explicit training and communication using short-time Fourier (STF) basis functions, that serve as approximate eigen-functions for such channels. Requirements on coherence time in existing works are naturally replaced with requirements on the time-frequency coherence dimension in STF signaling. Motivated by recent measurement campaigns, we propose a sparse multipath channel model in which the coherence dimension *naturally* scales with signal-space dimensions. Sparsity in the delay-Doppler domain affords two important benefits that have not been recognized thus far: 1) The coherence time requirement necessary to achieve an operational coherence level is dramatically reduced by exploiting sparsity in the delay domain, and 2) Sparsity in the Doppler domain can be used to achieve any operational level of coherence by appropriately scaling the signaling duration as a function of signaling bandwidth.

1 Introduction

The emergence of ultra-wideband radio and sensor networks has led to renewed interest in achieving coherent capacity in the wideband (or low SNR) regime. The coherent capacity of a channel is the maximum information rate that is achievable with arbitrary reliability assuming perfect CSI at the receiver. Coherent capacity may not be achievable in a practical communication system, either because the available energy is too small to learn the channel perfectly or because the fading is too fast. Therefore many recent works have focussed attention on achieving capacity in realistic communication scenarios where partial CSI is available at the receiver.

The seminal work by Verdu [1] has shown that the minimum energy per bit necessary for reliable communication, $\frac{E_b}{N_o}_{\min}$, and the wideband slope, S_0, are the two most important figures of merit to characterize the spectral efficiency in the wideband regime. A signaling scheme that

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achieves $\frac{E_b}{N_o}_{\min}$ is termed first-order optimal and one that achieves S_0 as well is termed second-order optimal. It is shown in [1] that when perfect CSI is available at the receiver (coherent setting), QPSK signaling achieves both the first and second-order optimality conditions in a fading channel under certain relaxed assumptions. If no CSI is available at the receiver (non-coherent setting), it is shown that *flashy* signaling is necessary and sufficient to achieve the first-order optimality condition. However, a flashy signaling scheme, besides having a peak-to-average ratio that tends to ∞ (and hence practically unrealizable), also results in the second derivative of capacity converging to $-\infty$ at zero SNR, leading to $S_0 = 0$.

This apparent sharp cut-off in the peak-to-average ratio of the capacity achieving signaling schemes between the coherent and non-coherent extremes is partly resolved by Zheng *et al.* who study explicit training-based and implicit channel-learning/communication schemes by relating the coherence time of the channel to the transmitted power per degree of freedom, SNR, (which is defined as $\mathsf{SNR} = \frac{P}{W}$) [2]. They show that capacities intermediate between the coherent and the non-coherent extremes can be achieved for an appropriate scaling of the coherence time with SNR. However [2] provides no physical basis/mechanism that would lead to such a scaling of coherence time with SNR.

In this paper, we first extend the results on capacity of training-based schemes in [2, 3] to doubly-dispersive underspread channels using an orthogonal short-time Fourier (STF) signaling scheme [4, 5]. There are three fundamental contributions of this paper relative to earlier works [2, 3]: 1) it extends the notion of coherence time to that of time-frequency coherence dimension via STF signaling and shows that sparsity of propagation paths in physical wideband channels (see, e.g., [6]) provides a natural mechanism for the scaling of coherence time and bandwidth with the signaling duration and bandwidth, respectively, 2) it shows that coherence requirements (to achieve an operational level of coherence) for sparser channels are dramatically weaker than that for rich multipath channels, and 3) these weaker coherence requirements of sparse channels can be achieved by communicating over longer signaling durations.

The time-frequency coherence dimension is inversely related to the delay-Doppler diversity [7] in the doubly-dispersive channel that is revealed by a canonical decomposition of the channel in terms of resolvable paths in delay-Doppler [8]. In contrast to [2] and [3], which show that the coherence time of the channel has to scale (with SNR) at a particular rate to achieve coherent capacity, we show that the time-frequency coherence dimension of the channel, defined as the product of the coherence time and coherence bandwidth, should scale at the corresponding rates to achieve coherent capacity. This simple observation has far-reaching consequences. In particular, in the case of sparse physical wideband channels in which the delay and Doppler diversities scale at a sub-linear rate with bandwidth and signaling duration, respectively, the sparsity in the delay domain can be exploited to lower the coherence time requirement while the sparsity in Doppler domain can be used to reduce the signaling duration to achieve an operational coherence level.

This paper is organized as follows. The system setup, including the channel model and training-based STF signaling scheme, is described in Section 2. The main result concerning the achievability of coherent capacity in the wideband limit is presented in Section 3. Section 4 concludes the paper with a discussion of the results in this paper.

2 System Setup

2.1 Sparse Channel Modeling

We consider a single-user single-antenna communication system in complex baseband

$$y(t) = \int_0^{T_m} \int_{-\frac{W_d}{2}}^{\frac{W_d}{2}} h(\tau, \nu) x(t - \tau) e^{j2\pi\nu t} \, \mathrm{d}\nu \, \mathrm{d}\tau + w(t) \quad (1)$$

where the channel is characterized by the delay-Doppler spreading function, $h(\tau, \nu)$, and $x(t)$, $y(t)$ and $w(t)$ represent the transmitted, received and additive white Gaussian noise waveforms, respectively. T_m and W_d represent the delay and Doppler spreads produced by the channel. We assume an underspread channel, $T_m W_d < 1$, which is valid for most radio channels. A physical discrete multipath channel can be modeled as

$$h(\tau, \nu) = \sum_n \beta_n \delta(\tau - \tau_n) \delta(\nu - \nu_n) \ , \ y(t) = \sum_n \beta_n x(t - \tau_n) e^{j2\pi\nu_n t} + w(t) \quad (2)$$

where β_n, $\tau_n \in [0, T_m]$ and $\nu_n \in [-W_d/2, W_d/2]$ denote the complex path gain, delay and Doppler shift associated with the n-th path. For a signaling duration T and bandwidth W, the

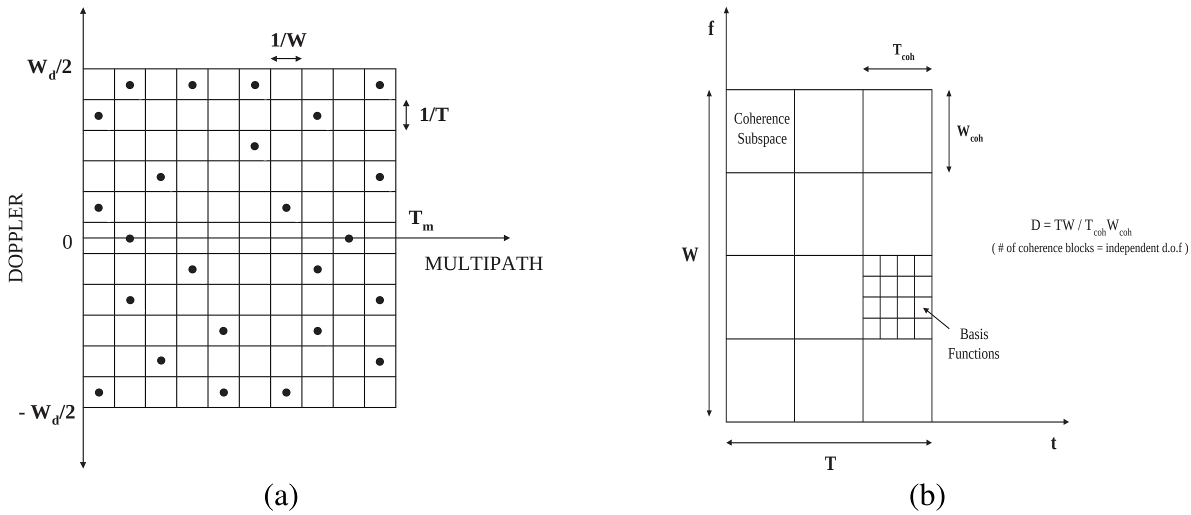

Figure 1: *(a) Delay-doppler sampling commensurate with signaling duration and bandwidth. (b) Time-frequency coherence subspaces in short-time Fourier signaling.*

channel admits the following decomposition [7, 8] illustrated in Fig. 1(a)

$$y(t) = \sum_{\ell=0}^{\lceil T_m W \rceil} \sum_{m=-\lceil TW_d/2 \rceil}^{\lceil TW_d/2 \rceil} h_{\ell,m} x(t - \ell/W) e^{j2\pi mt/T} \ , \ h_{l,m} \approx \sum_{n \in A_{l,m}} \beta_n \ , \quad (3)$$

where $A_{l,m} = \{n : \ell/W - 1/2W < \tau_n \leq \ell/W + 1/2W, m/T - 1/2T < \nu_n \leq m/T + 1/2T\}$ is the set of all paths whose delays and Doppler shifts lie within the (ℓ, m)-th delay-Doppler resolution bin. The number of resolvable paths signifies the delay-Doppler diversity, D, afforded by the channel (the number of statistically independent degrees of freedom (DoF))

$$D = D_T D_W \leq D_{max} = D_{T,max} D_{W,max} \ ; \ D_{T,max} = \lceil TW_d \rceil \ , \ D_{W,max} = \lceil T_m W \rceil \quad (4)$$

where $D_{T,max}$ denotes the maximum number of resolvable Doppler shifts (maximum Doppler/time diversity) and $D_{W,max}$ denotes maximum number of resolvable delays (maximum delay/frequency diversity). Both $D_{T,max}$ and $D_{W,max}$ increase linearly with T and W, respectively, representing a rich multipath environment in which each delay-Doppler resolution bin in Fig. 1(a) is populated with a path. On the other hand, as illustrated by the dotted resolution bins in Fig. 1(a), physical multipath channels get sparser with increasing W due to fewer than $D_{W,max}$ resolvable delays (see, e.g., [6] for experimental evidence) and with increasing T due to fewer than $D_{T,max}$ resolvable Doppler shifts. We model such sparse multipath channels with sub-linear scaling in D:

$$D_T \sim (TW_d)^{\delta_1} \ , \ D_W \sim (T_m W)^{\delta_2} \ , \ \delta_1, \delta_2 \in [0,1] \tag{5}$$

where the smaller the value of δ the slower (sparser) the growth in the resolvable paths in the corresponding domain. Note that this also implies that the DoF (delay-Doppler diversity), $D = D_T D_W$, scales sub-linearly with the number of signal space dimensions TW.

2.2 Orthogonal Short-Time Fourier Signaling

In this paper, we consider signaling over an orthonormal short-time Fourier (STF) basis [4, 5] that naturally relates delay-Doppler diversity to coherence in time-frequency. An orthogonal STF basis for the signal space is generated from a fixed prototype waveform $g(t)$ via time and frequency shifts: $\phi_{lm}(t) = g(t - lT_o)e^{j2\pi W_o t}$, where $T_o W_o = 1$ [4, 5]. For sufficiently underspread channels, T_o and W_o can be matched to T_m and W_d so that the STF basis waveforms serve as approximate eigenfunctions of the channel [4, 5]. Thus, representing (1) with respect to the STF basis functions results in a TW-dimensional matrix system equation

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \tag{6}$$

in which the $TW \times TW$ channel matrix $\mathbf{H}$ is diagonal

$$\mathbf{H} = \text{diag}\Big[\underbrace{h_{1,1}\cdots h_{1,N_{coh}}}_{\text{Subspace 1}}, \underbrace{h_{2,1}\cdots h_{2,N_{coh}}}_{\text{Subspace 2}} \cdots \underbrace{h_{D,1}\cdots h_{D,N_{coh}}}_{\text{Subspace } D}\Big] . \tag{7}$$

The diagonal entries of $\mathbf{H}$ also admit an intuitive block fading interpretation in terms of *time-frequency coherence subspaces* [5] illustrated in Fig. 1(b): the signal space is partitioned as $TW = N_{coh}D$ where D represents the number of statistically independent TF coherence subspaces (delay-Doppler diversity; see (4)) and N_{coh} represents the dimension of each coherence subspace. In the block fading model, the channel coefficients over the i-th coherence subspace $h_{i,1}\cdots h_{i,N_{coh}}$ are assumed to be identical, h_i, and the channel coefficients over the different coherence subspaces are assumed to be i.i.d. zero-mean Gaussian random variables (Rayleigh fading). The variance of the channel coefficients is denoted by $\alpha^2 = \mathbf{E}[|h_i|^2] = \sum_n \mathbf{E}[|\beta_n|^2]$. The dimension of each coherence subspace is given by

$$N_{coh} = T_{coh}W_{coh} = \frac{T}{D_T}\cdot\frac{W}{D_W} = \frac{T^{1-\delta_1}}{W_d^{\delta_1}}\frac{W^{1-\delta_2}}{T_m^{\delta_2}} \geq \left\lceil \frac{1}{T_m W_d} \right\rceil = N_{coh,min} \tag{8}$$

where $T_{coh} = T^{1-\delta_1}/W_d^{\delta_1}$ is the *coherence time* and $W_{coh} = W^{1-\delta_2}/T_m^{\delta_2}$ is the *coherence bandwidth* of the channel (see Fig. 1(b)). Note that $\delta_1 = \delta_2 = 1$ corresponds to a rich multipath channel in which $N_{coh} = N_{coh,min} = 1/(T_m W_d)$ is fixed and $D = D_{max}$ increases linearly with TW. This is assumed in all existing works. In contrast, for sparse channels ($\delta_i \in (0,1)$), both N_{coh} and D *increase sub-linearly* with T and W. As we show next, sparsity has far-reaching consequences in achieving near-coherent performance in the wideband regime.

2.3 Problem Formulation

In this paper, we use the block fading model induced by STF signaling to study the impact of time-frequency coherence on achieving coherent capacity in sparse multipath channels in the wideband/low-SNR regime. We say that a training scheme achieves an operational coherence level of ϵ if the sub-linear term of the average mutual information is $\mathcal{O}\left(\mathsf{SNR}^{1+\epsilon}\right)$. We assume a scaling in coherence dimension with SNR of the form $N_{coh} = N = \frac{k}{\mathsf{SNR}^{\mu}}$, $\mu > 0$ and quantify the coherence cost imposed by the channel (the value of μ) so that a training-based communication scheme achieves an operational coherence level of ϵ. We also assume that both the transmitter and receiver have knowledge of channel statistics, which is reasonable since channel statistics change over time-frames which are significantly longer than the coherence time. The knowledge of channel statistics (particularly the values of D_T and D_W) aids in the design of training and communication schemes that make efficient use of the signal space dimensions.

We now describe the training-based communication scheme, adapted from [2], suitable to STF signaling. The total energy available for training and communication is PT, of which a fraction η is used for training and the remaining fraction $(1-\eta)$ is used for communication. Since the quality of the channel estimate over one coherence subspace depends only on the training energy, our scheme uses one signal space dimension in each coherence subspace for training and the remaining $N-1$ for communication. The training energy per coherence subspace, E_{tr}, is then given by $E_{tr} = \frac{\eta TP}{D}$ and the communication energy per coherence subspace is given by $\frac{(1-\eta)TP}{(N-1)D}$. The following describes the training in the STF system:

$$\begin{aligned} y_{t,f} &= \sqrt{E_{tr}}\ h_{t,f}x_{t,f} + w_{t,f} = \sqrt{E_{tr}\alpha^2}\ g_{t,f}x_{t,f} + w_{t,f}\ , \\ & t = (i-1)\frac{T_{coh}}{T_o} + 1\ ,\ f = (j-1)\frac{W_{coh}}{W_o} + 1, \\ & i = 1, \cdots, D_T\ ,\ j = 1, \cdots, D_W \end{aligned} \tag{9}$$

where $\mathbf{E}\left[|x_{t,f}|^2\right] = 1$, $\mathbf{E}\left[|h_{t,f}|^2\right] = \alpha^2$ and $\mathbf{E}\left[|g_{t,f}|^2\right] = 1$. The minimum mean squared error (MMSE) estimate of $g_{t,f}$ is given by $\widehat{g_{t,f}} = \frac{\sqrt{E_{tr}\alpha^2}}{1+E_{tr}\alpha^2}\ y_{t,f}x^*_{t,f}$ with the MSE given by $\frac{1}{1+E_{tr}\alpha^2}$.

3 Capacity of the Training-Based STF System

We first characterize the coherent capacity of the single antenna wideband channel.

Proposition 1. *For all* $b \in (0,1)$ *and* $\mathsf{SNR} = \frac{P}{W}$ *such that* $\mathsf{SNR} < \frac{(1-b)}{b\ \alpha^2}$*, the coherent capacity per dimension,* C_{coh} *(in bps/Hz), satisfies*

$$\log_2(e)\left(\alpha^2\ \mathsf{SNR} - \alpha^4\ \mathsf{SNR}^2\right) \le C_{coh} \le \log_2(e)\left(\alpha^2\ \mathsf{SNR} - \frac{b}{2}\cdot\alpha^4\ \mathsf{SNR}^2\right). \tag{10}$$

Moreover at low SNR*,* $C_{coh} = \log_2(e)\left(\alpha^2\ \mathsf{SNR} - \alpha^4\ \mathsf{SNR}^2\right)$.

In particular, Proposition 1 shows that the minimum energy per bit necessary for reliable communication is given by $\frac{E_b}{N_o}_{\min} = \frac{\log_e(2)}{\alpha^2}$. The following lemma provides a lower bound to the capacity of the channel based on the training-based scheme.

Lemma 1. *The coherent capacity of the channel is lower bounded by*

$$I_1 = \left(1 - \frac{1}{N}\right)\cdot\mathbf{E}\left[\log_2\left(1 + \beta\alpha^2\,|\widehat{g}|^2\right)\right] \tag{11}$$

where $\beta = \frac{(1-\eta)\ (D+\alpha^2\eta\ TP)\ TP}{D[(N-1)(D+\alpha^2\eta\ TP)+\alpha^2(1-\eta)\ TP]}$ *and* $\widehat{g}$ *is a zero mean random variable with* $\mathbf{E}\left[|\widehat{g}|^2\right] = \sigma^2 = \frac{\alpha^2\eta\ TP}{D+\alpha^2\eta\ TP}$.

Proof. The coherent capacity of the channel is lower bounded by the average mutual information of the training-based communication scheme described in Section 2. Representing the $(N-1)D$-dimensional communication sub-channel of the matrix channel in (6) by $\mathbf{H}$ for simplicity and using a zero-mean Gaussian input with covariance matrix $\mathbf{Q} = \frac{\mathrm{Tr}(\mathbf{Q})}{(N-1)D} I_{(N-1)D}$ where $\mathrm{Tr}\left(\mathbf{Q}\right) = (1-\eta)\ TP$, we have the following lower bound to C_{coh} [10]

$$C_{coh} \geq I_1 = \frac{1}{ND} \cdot \mathbf{E}\left[\log_2 \det\left(I_{(N-1)D} + \hat{\mathbf{H}}\mathbf{Q}\hat{\mathbf{H}}^H\left(I_{(N-1)D} + \Sigma_{\Delta\mathbf{x}}\right)^{-1}\right)\right] \tag{12}$$

where $\hat{\mathbf{H}}$ is the $(N-1)D$-dimensional diagonal matrix of channel estimates and Δ is the error matrix $\mathbf{H} - \hat{\mathbf{H}}$. We note that $\Sigma_{\Delta\mathbf{x}} = \frac{\alpha^2}{1+\alpha^2 E_{tr}} \cdot \frac{\mathrm{Tr}(\mathbf{Q})}{(N-1)D} I_{(N-1)D}$ since the diagonal entries h_i are identically distributed. We thus have

$$I_1 = \frac{1}{ND} \cdot \mathbf{E}\left[\log_2 \det\left(I_{(N-1)D} + \beta\hat{\mathbf{H}}\hat{\mathbf{H}}^H\right)\right] = \left(\frac{N-1}{ND}\right) \cdot \sum_{i=1}^{D} \mathbf{E}\left[\log_2\left(1 + \beta\alpha^2\left|\widehat{\mathbf{g}}_i\right|^2\right)\right] \tag{13}$$

where β is as in the statement of the lemma and $\widehat{g}_i$ are i.i.d. zero mean random variables with $\mathbf{E}\left[|\widehat{g}_i|^2\right] = \frac{\alpha^2\eta\ TP}{D+\alpha^2\eta\ TP}$. This proves the lemma. □

We now provide a reverse Jensen's inequality-type lower bound to I_1.

Lemma 2. *A more tractable lower bound to* C_{coh} *is*

$$I_2 = \left(1 - \frac{1}{N}\right) \cdot \left[\log_2\left(1 + \beta\alpha^2\sigma^2\right) - 2\log_2(e)\beta^2\alpha^4\sigma^4\right] \tag{14}$$

where β *and* σ^2 *are as in Lemma 1.*

Proof. For any positive random variable z, we have

$$\begin{aligned}\mathbf{E}\left[\log_2\left(1+z\right)\right] - \log_2\left(1+\mathbf{E}\left[z\right]\right) &\overset{(a)}{\geq} \log_2(e)\left(\mathbf{E}\left[\frac{z}{1+z}\right] - \mathbf{E}[z]\right)\\ &\overset{(b)}{\geq} \log_2(e)\left(\mathbf{E}\left[z(1-z)\right] - \mathbf{E}[z]\right) = -\log_2(e)\ \mathbf{E}[z^2]\end{aligned}$$

where (a) follows from $\frac{z}{1+z} \leq \log_e(1+z) \leq z$ and (b) follows from the fact that $\frac{1}{1+z} \geq 1-z$. Using the above and the Gaussianity of the channel estimates in (11) completes the proof. □

We now optimize over the fraction of energy spent for training, η, to maximize the lower bound I_2. Our result is stated in the following two propositions.

Proposition 2. *The* η *that optimizes* I_2 *given by (14) satisfies* $\frac{\mathrm{d}y}{\mathrm{d}\eta} = 0$ *where* $y(\eta) = y = \beta\sigma^2$ *and* β *and* σ^2 *are as in Lemma 1.*

Proof. The derivative of I_2 with respect to η can be written as

$$\frac{\mathrm{d}I_2}{\mathrm{d}\eta} = \frac{c_1}{1+\alpha^2 y}\frac{\mathrm{d}y}{\mathrm{d}\eta}\ \left(2(\sqrt{2}-1)\alpha^2 y + 1\right) \cdot \left(2(\sqrt{2}+1)\alpha^2 y - 1\right) \tag{15}$$

where c_1 is a constant independent of η and SNR. We now show that $\max_\eta \alpha^2 y \to 0$ as $\mathsf{SNR} \to 0$, which implies that the optimal η should satisfy $\frac{\mathrm{d}y}{\mathrm{d}\eta} = 0$. The quantity $\alpha^2 y$ can be written as $\frac{(\alpha^2 TP)^2}{D} \cdot \frac{\eta(1-\eta)}{(N-1)(D+\alpha^2\ \eta\ TP)+\alpha^2\ (1-\eta)\ TP}$. It is easy to check that the η that maximizes $\alpha^2 y$ is $\eta^\star = \frac{\alpha^2 TP+(N-1)D}{(N-2)\alpha^2 TP} \cdot \left[\sqrt{1+\frac{\alpha^2 TP(N-2)}{\alpha^2 TP+(N-1)D}} - 1\right]$. After elementary algebra, $K \doteq \alpha^2 y\Big|_{\eta^\star} = \frac{\alpha^2 TP+(N-1)D}{D(N-2)^2} \cdot \left[\sqrt{1+\frac{\alpha^2 TP(N-2)}{\alpha^2 TP+(N-1)D}} - 1\right]^2$. Upper bounding and approximating K we have

$$K = \max_\eta \alpha^2 y \leq \frac{2\max(NTP,\ ND)}{N^2 D} \overset{(c)}{=} 2\max\left(\frac{P}{W}, \frac{1}{N}\right) = 2\max\left(\mathsf{SNR}, \mathsf{SNR}^\mu\right) \qquad (16)$$

where (c) follows from $ND = TW$. Since we are studying the achievability of coherent capacity as $\mathsf{SNR} \to 0$, we have $K = \max_\eta \alpha^2 y = \max_\eta \alpha^2 \beta\sigma^2 \to 0$. □

Proposition 3. *The $\eta^\star$ from Proposition 2 optimizes I_2 and the tightest lower bound for I_2 is*

$$I_2 = \left(1 - \frac{1}{N}\right) \cdot \left[\log_2(1+K) - 2\log_2(e)K^2\right] \qquad (17)$$

where K is as in Proposition 2.

Proof. It is easy to see that the optimizing η is a root of the quadratic $\eta^2\left(TP\alpha^2(N-2)\right) + 2\eta\left(TP\alpha^2 + (N-1)D\right) - \left(TP\alpha^2 + (N-1)D\right) = 0$ and is precisely $\eta^\star$ of Proposition 2. Using this value of $\eta^\star$ in (14) proves the proposition. □

The next result characterizes the coherence cost on the channel so that any operational coherence level (in particular, the first and second-order optimality conditions) can be achieved. Specifically, we assume that $N_{coh} = N = \frac{k}{\mathsf{SNR}^\mu}, k = \mathcal{O}(1)$ and characterize μ such that $I \geq \mathsf{SNR} - \mathcal{O}\left(\mathsf{SNR}^{1+\epsilon}\right)$.

Theorem 1. *The average mutual information of the training-based scheme satisfies*

$$I \geq \mathsf{SNR} - \mathcal{O}\left(\mathsf{SNR}^{1+\epsilon}\right) \qquad (18)$$

if and only if $N_{coh} = \frac{k}{\mathsf{SNR}^\mu}$ for $\mu \geq 1 + 2\epsilon$. In particular, the first and second order optimality conditions at low SNR are met if and only if $\mu > 1$ and $\mu > 3$, respectively.

Proof. See Appendix B. □

4 Discussion

We first interpret our results in the context of existing works that assume rich multipath: both delay and Doppler diversity scale linearly with W and T, respectively (W_{coh} and T_{coh} are fixed). Under this assumption, our results reduce to that in [2] and [3] since $T_{coh} = \mathcal{O}(N_{coh})$. From Theorem 1, we see that to achieve an operational coherence level ϵ, $\frac{1}{T_m W_d} = T_{coh} W_{coh} = N_{coh} = \mathcal{O}\left(W^{1+2\epsilon}\right)$, or in other words, the channel has to become more and more underspread as W increases. Such a restriction on the channel is physically impossible to meet. The contribution of this paper, relative to earlier works [2] and [3], is to study the impact of sparsity of propagation paths in physical wideband channels on the time-frequency coherence requirements to achieve coherent capacity. As discussed in Section 2.1, in sparse wideband channels, D_W and W_{coh} increase sub-linearly with W. Furthermore, unlike existing works, our results also explicitly account for Doppler diversity (D_T and T_{coh} increase sub-linearly with T) since STF signaling involves coding over multiple coherence times.

In contrast to [2, 3], Theorem 1 shows that the requirement on T_{coh} is now the requirement on time-frequency coherence dimension $N_{coh} = T_{coh}W_{coh}$. Thus, unlike [2] and [3], the coherence cost is shared in both time and frequency and as a result the T_{coh} requirement can be significantly weakened in the case of sub-linear (sparse) scaling in the number of resolvable paths in the delay domain. If the delay diversity is known to scale as $D_W = \mathcal{O}\left(W^{\delta_2}\right)$, the T_{coh} requirement can be reduced to $T_{coh} = N_{coh}/W_{coh} = \mathcal{O}\left(W^{2\epsilon+\delta_2}\right)$ to achieve an operational coherence level of ϵ, as defined in Section 2.3. Using $\epsilon = \frac{1}{2}$, which will result in an operational coherence level corresponding to a sub-linear term of $\mathsf{SNR}^{1.5}$ for the training scheme, and $\delta_2 = \frac{1}{2}$, we get $T_{coh} = \mathcal{O}(W^{\frac{3}{2}})$, a new and less stringent scaling law. Fig. 2(a) shows the weaker coherence time requirement for sparser channels for the following parameters: $T_m = 10^{-5}$ secs., $W_d = 50$ Hz, $W = 50$ MHz.

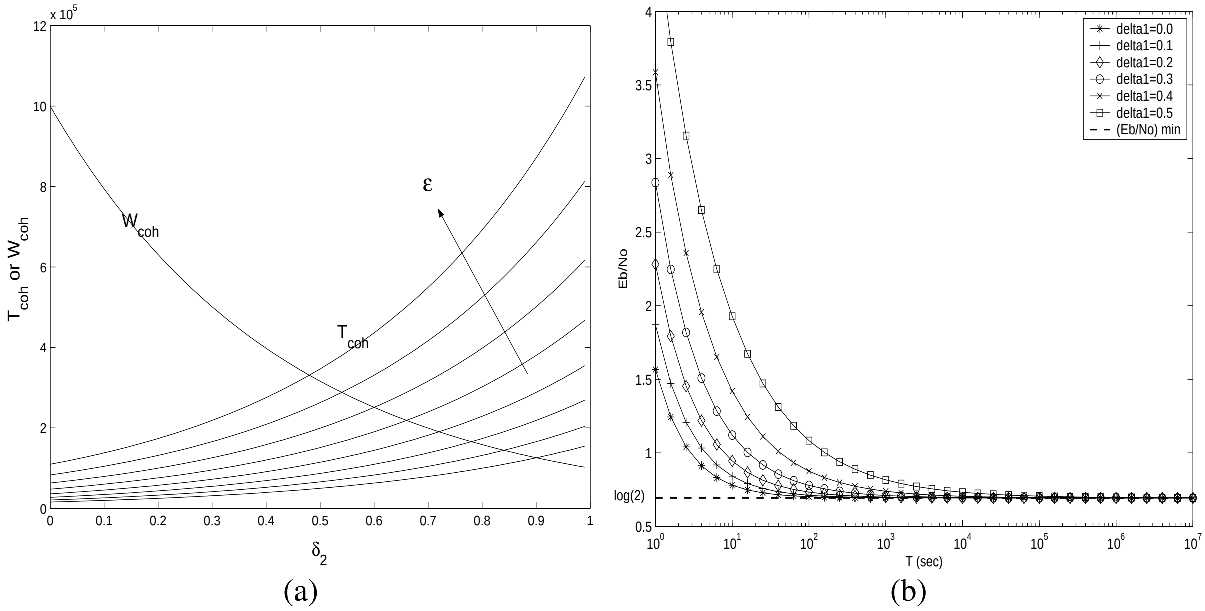

Figure 2: *(a) The variation of T_{coh} and W_{coh} as a function of delay sparsity. (b) $\frac{E_b}{N_0}$ vs. T for varying δ_1.*

Sparsity in the Doppler domain further relaxes the requirements on the channel. For a fixed large but finite bandwidth W so that we are in the wideband regime, if we know the scaling behavior of Doppler diversity (δ_1), our results show that the signaling duration T needs to only satisfy

$$\log(T) = \frac{1}{1-\delta_1}\log\left(W_d^{\delta_1}T_m^{\delta_2}\right) + \left(\frac{2\epsilon+\delta_2}{1-\delta_1}\right)\log(W) \tag{19}$$

to achieve an operational coherence level ϵ. Note that smaller δ_i imply a slower scaling of T with W. Conversely, for any system operating at a particular T and W, (19) can be used to determine the effective value of μ in the relationship $N_{coh} = \mathcal{O}(W^{\mu_{eff}})$, and ascertain the operational coherence level of the system, ϵ_{eff} (defined by $\epsilon_{eff} = \frac{\mu_{eff}-1}{2}$):

$$\mu_{eff} = \frac{\log(T/k)}{\log(W)}(1-\delta_1) + (1-\delta_2) \tag{20}$$

where $k = \left(T_m^{\delta_2}W_d^{\delta_1}\right)^{\frac{1}{1-\delta_1}}$. Note that $\mu_{eff} \to \infty$ as $T \to \infty$ for sparse channels, which implies that any operational level of coherence can be achieved by simply increasing T. This is in direct contrast to the case of rich multipath where the coherence requirement is independent of signaling duration. Fig. 2(b) illustrates the increase of μ_{eff} with T for sparse channels.

From (20), we see that for some sufficiently large value of T, the value of μ_{eff} crosses the threshold value of $\mu = 1$, necessary for first order optimality, and the corresponding energy per bit reduces to $\frac{E_b}{N_o}_{\min}$.

A Proof of Proposition 1

The coherent capacity per dimension is defined as

$$C_{coh} = \frac{\sup_{\mathbf{Q}:\ \mathrm{Tr}(\mathbf{Q}) \leq TP} \mathbf{E}\left[\log_2 \det\left(I_{ND} + \mathbf{H}\mathbf{Q}\mathbf{H}^H\right)\right]}{ND} \tag{21}$$

where the optimization is over the set of positive semi-definite transmit covariance matrices. The uniform power allocation $\mathbf{Q} = \frac{TP}{ND} I_{ND} = \mathsf{SNR}\ I_{ND}$ achieves this optimum and we have

$$C_{coh} = \frac{\sum_{i=1}^{D} \mathbf{E}\left[\log_2\left(1 + \frac{TP}{ND}|h_i|^2\right)\right]}{D} \overset{(a)}{=} \mathbf{E}\left[\log_2\left(1 + \mathsf{SNR}\,|h|^2\right)\right] \tag{22}$$

where (a) follows since $\{h_i\}$ are i.i.d. with h representing a generic random variable, $ND = TW$ and $\mathsf{SNR} = \frac{P}{W}$. The upper bound of the proposition follows from a combination of Jensen's inequality and the monotonicity of $\log_e(1+x) - x + \frac{bx^2}{2}$ under the imposed constraints. On the other hand, computing the expectation operation of (22) in closed form [9] and lower bounding using the Lebesgue Dominated Convergence Theorem yields the proposition.

B Proof of Theorem 1

For simplicity, we prove the theorem for the special case of second order optimality (or $\epsilon = 1$). The general case is a simple extension of the type of algebra done here. With $N = k\ \frac{1}{\mathsf{SNR}^{\mu}}$, we rewrite D as $D = \frac{TW\mathsf{SNR}^{\mu}}{k}$. We then write K as

$$\begin{aligned} K &= K_1 K_2\ ,\ K_1 = \frac{\mathsf{SNR}^{\mu}\ \left(k\alpha^2\ \mathsf{SNR} + k - \mathsf{SNR}^{\mu}\right)}{\left(k - 2\mathsf{SNR}^{\mu}\right)^2} \\ K_2 &= \left[\sqrt{1 + \frac{k\alpha^2\ \mathsf{SNR}^{1-\mu}\ \left(k - 2\mathsf{SNR}^{\mu}\right)}{k\alpha^2\ \mathsf{SNR} + k - \mathsf{SNR}^{\mu}}} - 1\right]^2 . \end{aligned} \tag{23}$$

We study the low SNR asymptotics of K for the following four cases – Case 1: $\mu = 1$, Case 2: $\mu \in (1, 3)$, Case 3: $\mu \geq 3$ and Case 4: $\mu < 1$.

Case 1: It is not difficult to check that

$$K_1 = \frac{\mathsf{SNR}}{k} + \mathcal{O}(\mathsf{SNR}^2)\ ,\ K_2 = \left(\sqrt{1 + k\ \alpha^2 + \mathcal{O}(\mathsf{SNR}) + \mathcal{O}(\mathsf{SNR}^2)} - 1\right)^2 = \mathcal{O}(1). \tag{24}$$

Using the above relationships in (14), we see that first and second order optimality conditions hold upto an order relationship. However, exact first order optimality is never possible in this setting.

Case 2: When $\mu \in (1,3)$, we have

$$K_1 = \frac{\mathsf{SNR}^{\mu}}{k} \sum_{i=\{0,1\}} \sum_{j=0}^{j=\infty} \mathcal{O}\left(\mathsf{SNR}^{i+j\mu}\right)$$

$$K_2 = \alpha^2\, k \cdot \frac{1}{\mathsf{SNR}^{\mu-1}} \left[1 + \frac{2}{k\alpha^2}\mathsf{SNR}^{\mu-1} - \frac{\mathsf{SNR}}{k} - 2\left(\frac{1}{k\alpha^2}\right)^{\frac{1}{2}} \mathsf{SNR}^{\frac{\mu-1}{2}} - \left(\frac{1}{k\alpha^2}\right)^{\frac{3}{2}} \mathsf{SNR}^{\frac{3\mu-3}{2}} + \left(\frac{1}{2k\alpha^2}\right)^{2} \mathsf{SNR}^{2\mu-2} \right] \tag{25}$$

which implies that one of the SNR^{μ}, $\mathsf{SNR}^{\frac{\mu+1}{2}}$, $\mathsf{SNR}^{\frac{3\mu-1}{2}}$, $\mathsf{SNR}^{2\mu-1}$ terms in K leads to failure of second order optimality condition.

Case 3: When $\mu \geq 3$, K_1 and K_2 are given by (25) and every vanishing term is of the form SNR or SNR^{ν} for some $\nu \geq 2$. Thus both first and second order optimality condition are met.

Case 4: When $\mu < 1$, K_1 is given by the same relationship as in (25). But for K_2 we have

$$K_2 = \left(\frac{k\alpha^2}{2}\, \mathsf{SNR}^{1-\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathcal{O}\left(\mathsf{SNR}^{i+j\mu}\right)\right)^2 . \tag{26}$$

This results in the failure of the first order optimality condition since the largest power of SNR in the Taylor's series expansion of I_2 is $\mathsf{SNR}^{2-\mu}$.

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